

An explicit formula for the deformation quantization of Lie bialgebras

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Abstract

A model of 3-dimensional topological quantum field theory is rigorously constructed. The results are applied to an explicit formula for deformation quantization of any finite-dimensional Lie bialgebra over the field of complex numbers. This gives an explicit construction of "quantum groups" from any Lie bialgebra, which was proven without explicit formulas in [EK].

Introduction

The most ideal goal of this paper would be a construction of an L_∞ -structure on the Gerstenhaber-Schack complex $K_{GS}^\bullet(A)$ of a (co)associative bialgebra A and a proof of its formality for $A = S(V^*)$, the free commutative cocommutative bialgebra of polynomial functions on a finite-dimensional vector space V . Below in the Introduction it is explained what part of this project is realized here.

First of all, let us recall the definitions. A (co)associative bialgebra A is a vector space endowed with two operations, $*$: $A^{\otimes 2} \rightarrow A$ and Δ : $A \rightarrow A^{\otimes 2}$, called the product and the coproduct, correspondingly. These operations should obey the following 3 axioms:

- (i) $(a * b) * c = a * (b * c)$ for any $a, b, c \in A$ (the associativity),
- (ii) $(\Delta \otimes 1) \circ \Delta(a) = (1 \otimes \Delta) \circ \Delta(a)$ for any $a \in A$ (the coassociativity),
- (iii) $\Delta(a * b) = \Delta(a) * \Delta(b)$ for any $a, b \in A$ (the compatibility)

(Here in the r.h.s. of (iii) the product $*$ on $A^{\otimes 2}$ is defined as $(a_1 \otimes a_2) * (b_1 \otimes b_2) = (a_1 * b_1) \otimes (a_2 * b_2)$).

Notice that a (co)associative bialgebra could not have the (co)unit and the antipode.

In the case of an associative algebra A there is the well-known construction of the Hochschild cohomological complex $\text{Hoch}^\bullet(A)$ and the Gerstenhaber bracket on it which makes $\text{Hoch}^\bullet(A)$ a dg Lie algebra. This dg Lie algebra plays a fundamental role in the deformation theory of associative algebras. Namely, the deformation functor associated

with this dg Lie algebra describes the deformations of the algebra A in the class of associative algebras (more precisely, it describes the deformations of the category of A -modules). Roughly, it means that for a cochain $\Psi \in \text{Hom}_{\mathbb{C}}(A^{\otimes 2}, A) = \text{Hoch}^1(A)$ the Maurer-Cartan equation

$$d\Psi + \frac{1}{2}[\Psi, \Psi] = 0 \quad (1)$$

in $\text{Hoch}^\bullet(A)$ is equivalent that the product $a \tilde{*} b = a * b + \Psi(a, b)$ is again associative (see, e.g. [K1] for details).

There is a complex which could be considered as analog of $\text{Hoch}^\bullet(A)$ in the case of (co)associative bialgebras, namely, the Gerstenhaber-Schack complex $K_{GS}^\bullet(A)$ [GS]. Recall that

$$K_{GS}^\bullet(A) = \bigoplus_{m, n \geq 1} \text{Hom}_{\mathbb{C}}(A^{\otimes m}, A^{\otimes n})[-m - n + 2] \quad (2)$$

(in the agreement with the usual notations, $(L^\bullet[s])^k = L^{s+k}$; in particular, if A has only degree 0, $\text{Hom}_{\mathbb{C}}(A^{\otimes m}, A^{\otimes n})[-m - n + 2]$ has degree $m + n - 2$). In particular, if A has only degree 0, $K_{GS}^1(A) = \text{Hom}(A^{\otimes 2}, A) \oplus \text{Hom}(A, A^{\otimes 2})$. In [GS], Gerstenhaber and Schack constructed a differential on $K_{GS}^\bullet(A)$ for any bialgebra A such that the first cohomology $H^1(K_{GS}^\bullet(A))$ is isomorphic to the infinitesimal deformations of the (co)associative bialgebra structure on A .

Even the first attempt to construct a Lie algebra structure on $K_{GS}^\bullet(A)$ which would describe the global deformations of (co)associative bialgebras via the Maurer-Cartan equation (1) fails. Indeed, consider $\Psi_1 + \Delta_1 \in \text{Hom}(A^{\otimes 2}, A) \oplus \text{Hom}(A, A^{\otimes 2}) = K_{GS}^1(A)$. The Maurer-Cartan equation for any possible bracket is *quadratic* in Ψ_1, Δ_1 , while the r.h.s. of the compatibility equation (iii) in the definition of (co)associative bialgebra above is of the 4th degree in Ψ_1, Δ_1 . Therefore, the best we can expect is the existence of an L_∞ -algebra structure on $K_{GS}^\bullet(A)$ with the first component equal to the Gerstenhaber-Schack differential. Recall that an L_∞ -algebra on a \mathbb{Z} -graded vector space L^\bullet is an odd vector field Q of degree +1 on the space $L^\bullet[1]$ such that $Q^2 = 0$. There exists a deformation theory associated with an L_∞ -algebra: the Maurer-Cartan equation (1) is replaced by the equation

$$\{\gamma \in L^1 \text{ such that } Q|_\gamma = 0\} \quad (3)$$

It would be very nice to construct such an odd vector field Q on $K_{GS}^\bullet(A)[1]$ such that the generalized Maurer-Cartan equation (3) describes exactly the (co)associative bialgebras. This problem still remains to be open.

If we would find an L_∞ -structure on $K_{GS}^\bullet(A)$ and prove explicitly the formality of it in the case when $A = S(V^*)$ (like it is done in [K1] in the case of associative algebras), we immediately would get an explicit construction of "quantum groups" from the infinitesimal datum—a Lie bialgebra structure on V . The theorem that any Lie bialgebra can be quantized was proven without explicit formulas in [EK].

In the present paper we find an explicit formula for the deformation quantization of any Lie bialgebra V . For this, we find an analog of the formality equation for $K_{GS}^\bullet(S(V^*))$, but with unknown L_∞ -structure. We derive this "formality" from the Stokes formula in some rigorously defined in the paper "3-dimensional topological quantum field theory". This "formality" equation is not the formality in the proper sense, because we do not know the L_∞ structure which is supposed to be formal, but it is enough to quantize any Lie bialgebra V .

To construct the "3-dimensional topological quantum field theory", we construct a compactification of the Kontsevich spaces $K(m, n)$ and of the extended Kontsevich spaces $K(m, n; s)$. In the case of associative algebras, the analog of $K(m, n)$ is the Stasheff space

$$\text{St}_n = \{(p_1, \dots, p_n) \in \mathbb{R}, p_i < p_j \text{ for } i < j\} / G^{(2)} \quad (4)$$

where the 2-dimensional group $G^{(2)}$ is the group of transformations

$$G^{(2)} = \{x \mapsto ax + b, a \in \mathbb{R}_+, b \in \mathbb{R}\} \quad (5)$$

Actually, the whole deformation theory of associative algebras is contained in the geometry of the Stasheff compactification $\overline{\text{St}}_n$ of the spaces St_n . Namely, the chain operad $M^\bullet = \bigoplus_{n \geq 2} C_\bullet(\overline{\text{St}}_n)$ in the realization by the Stasheff cells is free and is a minimal model of the operad Assoc of associative algebras. Then, the application of the Markl's construction from [M1] to this minimal model gives exactly the Hochschild complex with the Gerstenhaber bracket.

Unfortunately, we have not so nice description for our compactification $\overline{K(m, n)}$. We hope to understand the geometry of this compactification better in next papers. Now we have the description of all strata of codimension 1. Just notice here that it is *not* the CROC compactification from [Sh1] we suppose to be related with the unexisted theory of non-commutative deformations, but rather a Stasheff-type compactification. In particular, it is exactly the Stasheff compactification when $m = 1$ or $n = 1$.

In the case of associative algebras, an approach alternative to the explicit description of the chain operad as a minimal model (and which gives much more strong results) is the Kontsevich approach [K1]. Kontsevich "extends" the Stasheff space to a 2-dimensional configuration space $C_{m,n}$ and constructs its compactification $\overline{C}_{m,n}$ extending the Stasheff compactification. The space $C_{m,n}$ is the configuration space of m non-coinciding points at the upper half-plane and of n points at its boundary \mathbb{R} . Kontsevich gives in [K1] from this compactification a rigorous description of some particular case of the AKSZ model of topological quantum field theory on an open disk. For this, he constructs an appropriate "propagator" as a closed 1-form on $\overline{C}_{2,0}$. Within this approach, the Stokes formula gives "some relation" even if we would not know about the Gerstenhaber bracket. More precisely, the Gerstenhaber bracket is the only one which makes this equation to be a formality equation.

In this paper we follow the Kontsevich approach. We extend our compactification of the 1-dimensional configuration space $K(m, n)$ to a compactification of a 3-dimensional

configuration space $K(m, n; s)$ (which is the analog of the upper half-plane in the case of associative algebras). We construct all ingredients of a model of the 3-dimensional topological quantum field theory from this compactification. We construct a propagator as a closed 2-form *with singularities* on $\overline{K(0, 0; 2)}$ which degenerates to a closed 1-form in some limit. Then we associate some closed forms (with singularities) to the admissible graphs and apply the Stokes formula.

In this way, we replace the Gerstenhaber-Schack complex $K_{GS}^\bullet(A)$ to a homotopically equivalent complex $\tilde{K}_{GS}^\bullet(A)$. The definition of $\tilde{K}_{GS}^\bullet(A)$ itself depends on our compactification $\overline{K(m, n)}$.

The Stokes formula gives an equation which we would like to interpret as the "formality of L_∞ -morphism". The further research should shed some light to this claim.

The good problems (if you like, it is a definition of a good problem) in mathematics are valuable not only by themselves, but mostly by new ideas which appear when one tries to solve them. The author is sure that deformation theory of (co)associative bialgebras is a good problem. As far this problem is still not solved, one can expect that it will grow many further ideas. From this point of view the main idea of this paper is the introduction of the complex $B^{\bullet\bullet}(m, n)$. Probably, the problem studied here is the first problem where the introduction of it is really necessarily. We formulated Conjectures 1,2 in Section 1 which formalize which properties we need from this structure in a possible greater generality. We think that technically the introduction of this bicomplex is the main new thing invented in the paper.

1 The Kontsevich spaces $K(m, n)$, their Stasheff-type compactification, and the homotopical Gerstenhaber-Schack complex.

1.1 The compactification

First of all, recall the definition of the spaces $K(m, n)$ due to Maxim Kontsevich (see also [Sh]). We show in the sequel that these spaces and its compactification introduced below play a crucial role in the deformation theory of (co)associative bialgebras.

First define the space $\text{Conf}(m, n)$. By definition, $m, n \geq 1$, $m + n \geq 3$, and

$$\begin{aligned} \text{Conf}(m, n) = \{p_1, \dots, p_m \in \mathbb{R}^{(1)}, p_i < p_j \text{ for } i < j; \\ q_1, \dots, q_n \in \mathbb{R}^{(2)}, q_i < q_j \text{ for } i < j\} \end{aligned} \quad (6)$$

Here we denote by $\mathbb{R}^{(1)}$ and by $\mathbb{R}^{(2)}$ two different copies of a real line \mathbb{R} .

Next, define a 3-dimensional group G^3 acting on $\text{Conf}(m, n)$. This group is a semidirect product $G^3 = \mathbb{R}^2 \ltimes \mathbb{R}_+$ (here $\mathbb{R}_+ = \{x \in \mathbb{R}, x > 0\}$) with the following group law:

$$(a', b', \lambda') \circ (a, b, \lambda) = (\lambda'a + a', (\lambda')^{-1}b + b', \lambda\lambda') \quad (7)$$

where $a, b, a', b' \in \mathbb{R}, \lambda, \lambda' \in \mathbb{R}_+$. This group acts on the space $\text{Conf}(m, n)$ as

$$(a, b, \lambda) \cdot (p_1, \dots, p_m; q_1, \dots, q_n) = (\lambda p_1 + a, \dots, \lambda p_m + a; \lambda^{-1} q_1 + b, \dots, \lambda^{-1} q_n + b) \quad (8)$$

In other words, we have two independent shifts on $\mathbb{R}^{(1)}$ and $\mathbb{R}^{(2)}$ (by a and b), and \mathbb{R}_+ dilatates $\mathbb{R}^{(1)}$ by λ and dilatates $\mathbb{R}^{(2)}$ by λ^{-1} .

In our conditions $m, n \geq 1, m + n \geq 3$, the group G^3 acts on $\text{Conf}(m, n)$ freely. Denote by $K(m, n)$ the quotient-space. It is a smooth manifold of dimension $m + n - 3$.

We will need also the spaces $K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}$ introduced below. Recall here our definition of the space $K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}$ (generalizing the Kontsevich space $K(m, n)$) from [Sh]:

First define the space $\text{Conf}_{n_1, \dots, n_{\ell_2}}^{m_1, \dots, m_{\ell_1}}$. By definition,

$$\begin{aligned} \text{Conf}_{n_1, \dots, n_{\ell_2}}^{m_1, \dots, m_{\ell_1}} = \\ \{ p_1^1, \dots, p_{m_1}^1 \in \mathbb{R}^{(1,1)}, p_1^2, \dots, p_{m_2}^2 \in \mathbb{R}^{(1,2)}, \dots, p_1^{\ell_1}, \dots, p_{m_{\ell_1}}^{\ell_1} \in \mathbb{R}^{(1,\ell_1)}; \\ q_1^1, \dots, q_{n_1}^1 \in \mathbb{R}^{(2,1)}, q_1^2, \dots, q_{n_2}^2 \in \mathbb{R}^{(2,2)} \dots, q_1^{\ell_2}, \dots, q_{n_{\ell_2}}^{\ell_2} \in \mathbb{R}^{(2,\ell_2)} | \\ p_{i_1}^j < p_{i_2}^j \text{ for } i_1 < i_2; q_{i_1}^j < q_{i_2}^j \text{ for } i_1 < i_2 \} \quad (9) \end{aligned}$$

Here $\mathbb{R}^{(i,j)}$ are copies of the real line \mathbb{R} . Now we have an $\ell_1 + \ell_2 + 1$ -dimensional group $G^{\ell_1, \ell_2, 1}$ acting on $\text{Conf}_{n_1, \dots, n_{\ell_2}}^{m_1, \dots, m_{\ell_1}}$. It contains $\ell_1 + \ell_2$ independent shifts

$$p_i^j \mapsto p_i^j + a_j, i = 1, \dots, m_j, a_j \in \mathbb{R}; q_i^j \mapsto q_i^j + b_j, i = 1, \dots, n_j, b_j \in \mathbb{R}$$

and *one* dilatation

$$p_i^j \mapsto \lambda \cdot p_i^j \text{ for all } i, j; q_i^j \mapsto \lambda^{-1} \cdot q_i^j \text{ for all } i, j.$$

This group is isomorphic to $\mathbb{R}^{\ell_1 + \ell_2} \ltimes \mathbb{R}_+$. We say that the lines $\mathbb{R}^{(1,1)}, \mathbb{R}^{(1,2)}, \dots, \mathbb{R}^{(1,\ell_1)}$ (corresponding to the factor λ) are the lines of the first type, and the lines $\mathbb{R}^{(2,1)}, \mathbb{R}^{(2,2)}, \dots, \mathbb{R}^{(2,\ell_2)}$ (corresponding to the factor λ^{-1}) are the lines of the second type.

Denote

$$K_{n_1, \dots, n_{\ell_2}}^{m_1, \dots, m_{\ell_1}} = \text{Conf}_{n_1, \dots, n_{\ell_2}}^{m_1, \dots, m_{\ell_1}} / G^{\ell_1, \ell_2, 1} \quad (10)$$

We are going to construct a compactification $\overline{K(m, n)}$ of the space $K(m, n)$. First consider the simplest examples.

Example

Consider the case when (m, n) is $(1, 3)$ or $(3, 1)$. Then the spaces $K(1, 3)$ or $K(3, 1)$ are the Stasheff spaces with 3 points. We compactify them as the corresponding Stasheff polyhedra to the unit closed interval. More generally, the space $\overline{K(m, 1)}$ (or $\overline{K(1, m)}$) in the compactification defined below is the Stasheff polyhedron St_m with m points.

Example

Let $m = n = 2$. Then the space $K(2, 2)$ is 1-dimensional. It is easy to see that $(p_2 - p_1) \cdot (q_2 - q_1)$ is preserved by the action of G^3 , and it is the only invariant of the G^3 -action on $K(2, 2)$. Therefore, $K(2, 2) \simeq \mathbb{R}_+$. There are two "limit" configurations: $(p_2 - p_1) \cdot (q_2 - q_1) \rightarrow 0$ and $(p_2 - p_1) \cdot (q_2 - q_1) \rightarrow \infty$. Therefore, the compactification $\overline{K(2, 2)} \simeq [0, 1]$. See Figure 1 below:

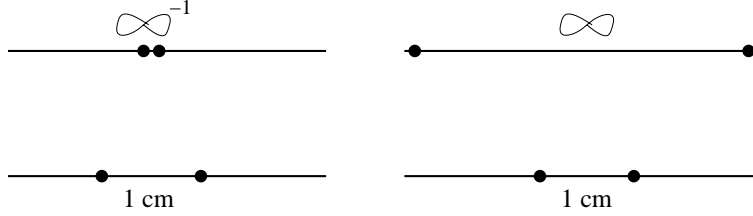


Figure 1: The two limit points in $\overline{K(2, 2)}$

Remark. The Kontsevich's insight when he introduced the spaces $K(m, n)$ was that the left picture in Figure 1 "should give" the l.h.s of the compatibility equation in the definition of (co)associative bialgebras, $\Delta(a * b)$, and the right picture in Figure 1 should give the r.h.s. $\Delta(a) * \Delta(b)$ of the compatibility equation. It will be more clear in the next Section when we define the compactification of the extended Kontsevich spaces $\overline{K(m, n; s)}$.

Now having in mind the two previous examples, we define the compactification in the general case.

Consider the set $\Sigma_{1,3}$ consisting from all possible samples of 1 point among the m points at the first line in $K(m, n)$ and of 3 points among the n points at the second line in $K(m, n)$. For any $\sigma \in \Sigma_{1,3}$ we have a map

$$r_\sigma: K(m, n) \rightarrow K(1, 3) \quad (11)$$

Analogously, we define the sets $\Sigma_{3,1}$ and $\Sigma_{2,2}$ as the sets of all samples of 3 points at the first line and 1 point at the second line, and of all samples of 2 points at each line, correspondingly. For any $\sigma \in \Sigma_{3,1}$ we have the map $r_\sigma: K(m, n) \rightarrow K(3, 1)$, and for any $\sigma \in \Sigma_{2,2}$ we have the map $r_\sigma: K(m, n) \rightarrow K(2, 2)$.

Now consider the map which is the product of all r_σ over all possible σ . It is a map

$$r = \prod_{\sigma \in \Sigma_{1,3} \sqcup \Sigma_{3,1} \sqcup \Sigma_{2,2}} r_\sigma: K(m, n) \rightarrow \prod_{\sigma \in \Sigma_{1,3}} K(1, 3)_\sigma \times \prod_{\sigma \in \Sigma_{3,1}} K(3, 1)_\sigma \times \prod_{\sigma \in \Sigma_{2,2}} K(2, 2)_\sigma \quad (12)$$

(Here the lower index σ in the r.h.s. just indicates the copy of the space associated with σ).

It is clear that the map r is *an imbedding*. Now we can compactify the image, $\prod_{\sigma \in \Sigma_{1,3}} K(1,3)_\sigma \times \prod_{\sigma \in \Sigma_{3,1}} K(3,1)_\sigma \times \prod_{\sigma \in \Sigma_{2,2}} K(2,2)_\sigma$, compactifying each factor as in the Examples above. We get an imbedding \bar{r} of $K(m,n)$ to a compact space (actually, a product of closed intervals). We define the compactification $\overline{K(m,n)}$ as the closure of the image of the map \bar{r} . The following lemma shows that the obtained compactification is of the Stasheff-type:

Lemma. (i) *The space $\overline{K(m,n)}$ is a manifold with corners,*

(ii) *any stratum of codimension 1 has the form as is shown in the Figure 2 below: Here in the Figure 2 m_0 points at the first line move close to each other in the scale $\frac{1}{\infty}$ and other points at this line are in finite distance from each other, and n_1 "external" points at the second line move infinitely far from each other with the scale ∞ (here this ∞ and ∞ in the fraction $\frac{1}{\infty}$ above are "the same"), and other "internal" $n_0 = n - n_1$ points are in finite distance from each other. Notice that after the application of the transformation $(0,0,\infty) \in "G(3)"$ the picture on the first line after the transformation becomes as the picture on the second line before the transformation, and wise versa. This stratum of codimension 1 is canonically isomorphic to $K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \times K_{1,1,\dots,1,m_0,1,\dots,1}^{n_1+1}$.*

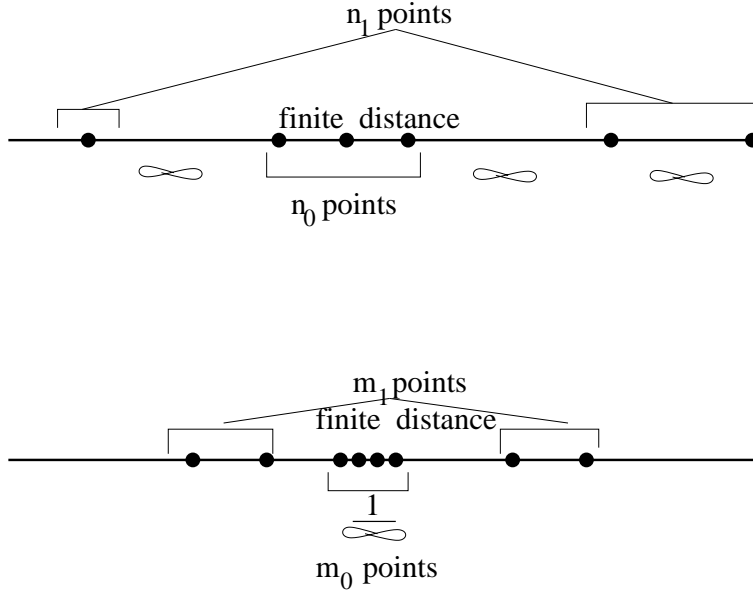


Figure 2: A typical stratum of codimension 1

Proof. We call a *4-point ratio* the image of any map r_σ defined above. It corresponds a real number to a sample of 1 point on the first line and 3 points on the second line, or to a sample of 3 points on the first line and 1 point on the second line, or to 2 points on each line.

Consider the maximal open strata in $\overline{K(m, n)}$ which is, by definition, the image of $K(m, n)$ under the imbedding r (before taking the closure). It is clear that the minimal number of 4-point ratios we should know to reconstruct the configuration which belongs to the maximal open stratum is exactly $m + n - 3$, the dimension of this open stratum. Moreover, we can introduce coordinates on it using the 4-point ratios.

Now define all k -dimensional strata in $\overline{K(m, n)}$ as the limit configurations which we can reconstruct from k of 4-point ratios ($k \leq m + n - 3$) and can not reconstruct by any $l < k$ 4-point ratios. It is in general nonconnected space, the connected components of which we call the strata of dimension k . We can use the 4-point ratios to introduce coordinates on it. It is clear that in this way we get a *manifold with corners*.

One easily sees that the strata like the stratum drawn in Figure 2 are uniquely defined by $m + n - 4$ 4-point ratios and not less of them, and they exhaust all such strata. \square

Now we are going to attach to the strata of codimension 1 some operations on the Gerstenhaber-Schack space $K_{GS}^\bullet(A)$ for any A .

1.2 The strata of codimension 1 and operations on $K_{GS}^\bullet(A)$

We first define the operations in a bit bigger generality than we will really need in the sequel. These operations is a particular case of Markl's fractions in [M2], and we independently introduced in the context of CROCs in [Sh1].

Let V be a vector space. Suppose we have

$$\Psi_1 \in \text{Hom}(V^{\otimes \ell_1}, V^{\otimes N_1}), \Psi_2 \in \text{Hom}(V^{\otimes \ell_1}, V^{\otimes N_2}), \dots, \Psi_{\ell_2} \in \text{Hom}(V^{\otimes \ell_1}, V^{\otimes N_{\ell_2}})$$

and

$$\Theta_1 \in \text{Hom}(V^{\otimes M_1}, V^{\otimes \ell_2}), \Theta_2 \in \text{Hom}(V^{\otimes M_2}, V^{\otimes \ell_2}), \dots, \Theta_{\ell_1} \in \text{Hom}(V^{\otimes M_{\ell_1}}, V^{\otimes \ell_2}),$$

we are going to define their composition which belongs to $\text{Hom}(V^{\otimes M_1 + \dots + M_{\ell_1}}, V^{\otimes N_1 + \dots + N_{\ell_2}})$. Denote $m = M_1 + \dots + M_{\ell_1}$, $n = N_1 + \dots + N_{\ell_2}$. The construction is as follows:

First define

$$\begin{aligned} F(v_1 \otimes \dots \otimes v_m) &:= \Theta_1(v_1 \otimes \dots \otimes v_{M_1}) \bigotimes \Theta_2(v_{M_1+1} \otimes \dots \otimes v_{M_1+M_2}) \bigotimes \dots \\ &\quad \bigotimes \Theta_{\ell_2}(v_{M_1+\dots+M_{\ell_1-1}+1} \otimes \dots \otimes v_{M_1+\dots+M_{\ell_1}}) \in V^{\otimes \ell_1 \ell_2} \end{aligned} \quad (13)$$

Now we apply $\{\Psi_i\}$'s to this element in $V^{\otimes \ell_1 \ell_2}$: we define an element $G: V^{\otimes \ell_1 \ell_2} \rightarrow V^{\otimes n}$ as follows:

$$G(v_1 \otimes v_2 \otimes \cdots \otimes v_{\ell_1 \ell_2}) := \Psi_1(v_1 \otimes v_{\ell_2+1} \otimes \cdots \otimes v_{\ell_2(\ell_1-1)+1}) \bigotimes \Psi_2(v_2 \otimes v_{\ell_2+2} \otimes \cdots \otimes v_{\ell_2(\ell_1-1)+2}) \bigotimes \cdots \bigotimes \Psi_{\ell_2}(v_{\ell_2} \otimes v_{2\ell_2} \otimes \cdots \otimes v_{\ell_2 \ell_1}) \in V^{\otimes n}. \quad (14)$$

Define now

$$Q(v_1 \otimes \cdots \otimes v_m) := G \circ F(v_1 \otimes \cdots \otimes v_m) \in V^{\otimes n} \quad (15)$$

By definition, the element Q is the composition $\frac{\Psi_1 \Psi_2 \cdots \Psi_{\ell_2}}{\Theta_1 \Theta_2 \cdots \Theta_{\ell_1}} \in \text{Hom}(V^{\otimes m}, V^{\otimes n})$.

In [Sh1], we introduced these compositions to associate operations on K_{GS}^\bullet with the strata of codimension 1 in $\overline{K(m, n)}_C$ in the CROC compactification. In the case of the compactification introduced here, we need only the following particular case of the construction above. Consider $\ell_1 = m_1 + 1$, $\ell_2 = n_1 + 1$ in the notations of Lemma above, and all from M_i 's are equal to 1 except a one which is equal to m_0 , and also all N_j 's are equal to 1 except one which is equal to n_0 . The reader could remind the Figure ... above.

Our (unrealized) goal is to construct an L_∞ structure on $K_{GS}^\bullet(V)$ (we can consider V as a (co)associative bialgebra with 0 product and 0 coproduct). According to this goal, we would like to consider the operations $\frac{\Psi_1 \Psi_2 \cdots \Psi_{\ell_2}}{\Theta_1 \Theta_2 \cdots \Theta_{\ell_1}}$ as candidates for components of an L_∞ structure on $K_{GS}^\bullet(V)$. The first thing we should check is the grading condition, and even here we runs to troubles. Indeed, an L_∞ operation is an operation of the form $\wedge^{m_1+n_1+2} \mathfrak{g}^\bullet \rightarrow \mathfrak{g}^\bullet[2 - (m_1 + 1) - (n_1 + 1)]$ where $\mathfrak{g}^\bullet = K_{GS}^\bullet(V)$ with the natural grading

$$\deg \text{Hom}(V^{\otimes m}, V^{\otimes n}) = m + n - 2 \quad (16)$$

Denote $A = (\sum_i \deg \Psi_i + \sum_j \deg \Theta_j) - m_1 - n_1$ and $B = m_0 + m_1 + n_0 + n_1 - 2$. We would have the correct L_∞ gradings iff $A = B$. Denote the defect $A - B$ by D . We have: $A = 2m_1n_1 + m_0 + n_0 - 2$, $D = 2m_1n_1 - m_1 - n_1$. We see that the defect $D = 0$ iff $(m_1, n_1) = (0, 0)$ or $(1, 1)$.

In the next Subsection we introduce a complex $\tilde{K}_{GS}^\bullet(V)$ quasi-isomorphic to the graded space $K_{GS}^\bullet(V)$ and redefine the operations $\frac{\Psi_1 \Psi_2 \cdots \Psi_{\ell_2}}{\Theta_1 \Theta_2 \cdots \Theta_{\ell_1}}$ on it such that they will be compatible with the L_∞ rule of degrees.

1.3 The homotopical Gerstenhaber-Schack space

For each (m, n) , $m, n \in \mathbb{Z}_+$, $m + n \geq 3$ introduce the bicomplex $B^{\bullet\bullet}(m, n)$ as follows:

Denote by $\Omega_+(\sigma)$ the de Rham complex of smooth differential forms on a stratum σ of the manifold with corners $\overline{K(m, n)}$ which can be continued to the *normalization* of the pair $(\sigma, \bar{\sigma})$. Here by the normalization we mean a compact space $\bar{\sigma}_{norm}$ with a projection $\bar{\sigma}_{norm} \rightarrow \bar{\sigma}$ which is a 1-1 map over the open stratum σ and which separates the points

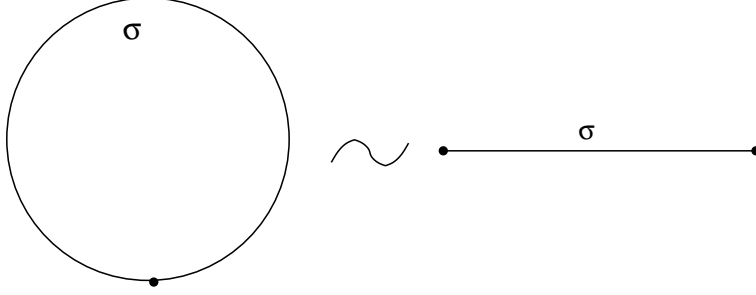


Figure 3: A stratification of the circle and its normalization

of the boundary $\bar{\sigma} \setminus \sigma$ which are limits of points which are far from each other on σ . For example, consider the stratification of the circle drawn in Figure 3 below. Here we have the two strata: the stratum σ and the point. The normalization of $\bar{\sigma}$ here is *not* the circle $\bar{\sigma}$, but an interval. Namely, we separate the two boundary points of $\bar{\sigma}$. In particular, $\Omega_+(\sigma)$ are *not* the forms on the circle, but the forms on the closed interval, they could have two different limits on the two boundary points. In the *chain* differential below we take the difference of the restrictions of a function (a 1-form) in the two boundary points in Figure 3.

We set:

$$B^{\bullet\bullet}(m, n) = \bigoplus_{\sigma \in \overline{K(m, n)}} \Omega_+(\sigma)[\dim \sigma] \quad (17)$$

We consider the two differentials of degree +1 on $B^{\bullet\bullet}(m, n)$: the first is the de Rham differential, and the second $-\partial$ is the chain differential in $\overline{K(m, n)}$ acting on σ 's *with the opposite sign*. The total cohomology of these two differentials is clearly 1-dimensional in the grading 0 and 0 otherwise. Denote by $[\omega]_\sigma$ a differential form ω on a stratum σ considered as an element in $B^{\bullet\bullet}(m, n)$. It is clear that $\deg[\omega]_\sigma = \deg \omega - \dim \sigma$. In particular, the bicomplex $B^{\bullet\bullet}(m, n)$ is $\mathbb{Z}_{\leq 0}$ -graded. We denote just by $[\omega]$ a differential form on the top degree open stratum.

Introduce

$$\tilde{K}_{GS}^\bullet(V) = \text{Hom}(V, V)[0] \oplus \bigoplus_{m+n \geq 3} \text{Hom}(V^{\otimes m}, V^{\otimes n}) \otimes B^{\bullet\bullet}(m, n) \quad (18)$$

It is clear that $\tilde{K}_{GS}^\bullet(V)$ is quasi-isomorphic to the graded space $K_{GS}^\bullet(V)$.

Now we are going to define "the right" operations $\frac{\Psi_1 \Psi_2 \dots \Psi_{\ell_2}}{\Theta_1 \Theta_2 \dots \Theta_{\ell_1}}$ on $\tilde{K}_{GS}^\bullet(V)$. First we consider the forms on the open top dimensional stratum.

Let $\tilde{\Psi}_i = \Psi_i \otimes [\omega_i]$ and $\tilde{\Theta}_j = \Theta_j \otimes [\omega'_j]$ where Ψ_i 's and Θ_j 's are as above. We consider the total degree: $\deg \tilde{\Psi}_i = \deg \Psi_i + \deg[\omega_i]$, and analogously for $\tilde{\Theta}_j$. Here all forms ω_i except the one are differential forms on the top degree stratum in $\overline{K(m_1 + 1, 1)}$ and the

remaining one is a form on the top dimensional stratum in $\overline{K(m_1 + 1, n_0)}$. Analogously, the all forms ω'_j 's except one are forms on the top dimensional stratum in $\overline{K(1, n_1 + 1)}$, and the remaining one is a form on the top dimensional stratum of $\overline{K(m_0, n_1 + 1)}$.

The operation $\frac{\Psi_1 \Psi_2 \dots \Psi_{\ell_2}}{\Theta_1 \Theta_2 \dots \Theta_{\ell_1}}$ itself is associated with a stratum σ of codimension 1 in $\overline{K(m, n)}$. The stratum σ is *canonically* isomorphic to $K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \times K_{1,1,\dots,1,m_0,1,\dots,1}^{n_1+1}$ (here in the upper index at the first factor there are n_1 of 1's and n_0 is at the s 'th place, and in the second factor in the lower index there are m_1 of 1's and m_0 is at the s' 'th place. First of all, construct a differential form ω_{tot} on this stratum σ of degree $\sum_i \deg \omega_i + \sum_j \deg \omega_j$, starting from the forms ω_i 's and ω'_j 's. The construction is as follows:

Consider the projections $p_1, \dots, p_{s-1}, p_{s+1}, \dots, p_{n_1+1}: K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \rightarrow K(m_1+1, 1)$ which is just the forgetting map which forgets the all upper points except the i 'th for p_i , and the forgetting map $p_s: K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \rightarrow K(m_1 + 1, n_0)$. Then take the wedge product $\Omega_1 = \wedge_{i=1}^{n_1+1} p_i^*(\omega_i)$ on $K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1}$. Analogously, define the projections p'_j and define the form $\Omega_2 = \wedge_{j=1}^{m_1+1} p'_j{}^* \omega'_j$ on $K_{1,1,\dots,1,m_0,1,\dots,1}^{n_1+1}$. Then consider the form $\omega := \Omega_1 \boxtimes \Omega_2$ on the product $K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \times K_{1,1,\dots,1,m_0,1,\dots,1}^{n_1+1}$. This is, by definition, the form ω defined on the open stratum $\sigma = K_{m_1+1}^{1,1,\dots,1,n_0,1,\dots,1} \times K_{1,1,\dots,1,m_0,1,\dots,1}^{n_1+1}$ of codimension 1 in $\overline{K(m, n)}$. It is clear, that $\deg \omega = \sum_i \deg \omega_i + \sum_j \deg \omega_j$.

Now define the composition $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ as

$$\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}} := \frac{\Psi_1 \Psi_2 \dots \Psi_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}} \otimes [\omega]_\sigma \quad (19)$$

It is clear that in this definition the L_∞ degree condition holds. Indeed, we need to prove that

$$\deg \tilde{\Psi}_1 + \dots + \deg \tilde{\Psi}_{n_1+1} + \deg \tilde{\Theta}_1 + \dots + \deg \tilde{\Theta}_{m_1+1} + (2 - (m_1 + n_1 + 2)) = \deg \frac{\Psi_1 \Psi_2 \dots \Psi_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}} + \deg [\omega]_\sigma \quad (20)$$

We have: $\deg \tilde{\Psi}_i = \deg \Psi_i + \deg \omega_i - (m_1 + 1 + 1 - 3) = 1 + \deg \omega_i$ for $i \neq s$ and $\deg \tilde{\Psi}_s = \deg \Psi_s + \deg \omega_s - (m_1 + 1 + n_0 - 3) = 1 + \deg \omega_s$. Analogously, $\deg \tilde{\Theta}_j = \deg \Theta_j + \deg \omega'_j - (n_1 + 1 + 1 - 3) = 1 + \deg \omega'_j$ for $j \neq s'$, and $\deg \tilde{\Theta}_{s'} = \deg \Theta_{s'} + \deg \omega'_{s'} - (n_1 + 1 + m_0 - 3) = 1 + \deg \omega'_{s'}$. Also, $\deg [\omega]_\sigma = \sum_{i=1}^{n_1+1} \deg \omega_i + \sum_{j=1}^{m_1+1} \deg \omega'_j - (m + n - 4)$ where $m = m_0 + m_1$ and $n = n_0 + n_1$ (here $m + n - 4$ is the dimension of the stratum

σ), and $\deg \frac{\Psi_1 \Psi_2 \dots \Psi_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}} = m + n - 2$. Then (20) reduces to

$$m_1 + n_1 + 2 - (m_1 + n_1) + \sum_i \deg \omega_i + \sum_j \deg \omega_j = m + n - 2 + \sum_i \deg \omega_i + \sum_j \deg \omega_j - (m + n - 4) \quad (21)$$

which surely holds.

Remark. It is clear now the origin of our problem with the "naive" definition of the composition on the level of $K_{GS}^\bullet(V)$. Namely, our Ψ and Θ there is identical to $\Psi \otimes [\omega_0]$ and $\Theta \otimes [\omega'_0]$ where ω_0 and ω'_0 are *top degree differential forms*. Then the resulting form ω on the stratum σ is 0 by dimensional reasons except few simplest cases (when $(m_1, n_1) = (0, 0)$ or $(1, 1)$). Then, the naively defined operation should be 0 from our point of view except these 2 cases. On the other hand, when the degrees of the forms ω_i 's and ω'_j 's are sufficiently small, the answer could be non-zero.

It remains to consider the general case, when the forms ω_i 's and ω'_j 's are defined on strata of any codimension (the previous case is the case of codimension 0).

1.3.1 The case of arbitrary strata

First of all, define a compactification of the space $K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}$. For each $1 \leq i \leq \ell_1$ and $1 \leq j \leq \ell_2$ we have a projection $p_{ij}: K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}} \rightarrow K(m_i, n_j)$ (we suppose that $m_i + n_j \geq 3$). Consider the map $\prod_{ij} p_{ij}: K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}} \rightarrow \prod_{ij} K(m_i, n_j)$. Clearly it is an embedding. Next, consider another embedding $i: \prod_{ij} K(m_i, n_j) \rightarrow \prod_{ij} \overline{K(m_i, n_j)}$. The composition $i \circ (\prod_{ij} p_{ij})$ is again an embedding. It embeds the open space $K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}$ to a compact space. Define the compactification $\overline{K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}}$ as the closure of the image of this embedding. It is a manifold with corners with the natural stratification, defined analogously with the stratification of $\overline{K(m, n)}$.

We give the following definition:

Definition. Let M_1, M_2 be two manifold with corners, and let $s: M_1 \rightarrow M_2$ be a continuous map. We say that s is a map of manifold with corners iff the preimage of each stratum in M_2 is a manifold with corners, and the manifold with corners M_1 is glued from these preimages (as a manifold with corners). Moreover, we demand that the restriction of s to the preimage of each stratum σ in M_2 is a trivial bundle, the fiber of which is a manifold with corners, and the total space of this bundle, as a manifold with corners, is the product of the open space σ with this fiber.

Conjecture 1. (i) the map p_{ij} defines a map of manifold with corners $\overline{p_{ij}}: \overline{K_{m_1, \dots, m_{\ell_1}}^{n_1, \dots, n_{\ell_2}}} \rightarrow \overline{K(m_i, n_j)}$,

(ii) the imbedding $i_\sigma: K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1} \times K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1} \rightarrow K(m, n)$ ($m = m_0 + m_1, n = n_0 + n_1$) is continued to a map of manifold with corners $\bar{i}_\sigma: \overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \times \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}} \rightarrow \overline{K(m, n)}$.

Define now the compositions $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ in the general case as follows:

Consider a stratum σ of some codimension in $\overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}}$. All images $\bar{p}_i: \overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \rightarrow \overline{K(m_1+1, N_i)}$ (here $N_i = 1$ for all i except $i = s$ for which $N_s = n_0$) are single strata in $\overline{K(m_1+1, N_i)}$ according to Conjecture 1(i). Define the image of σ with respect to the map \bar{p}_i by σ_i . As well, consider a stratum σ' of some codimension in $\overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}}$. Consider the maps $\bar{p}_j': \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}} \rightarrow \overline{K(M_j, n_1+1)}$. Denote the stratum in $\overline{K(M_j, n_1+1)}$ which is the image of σ' with respect to the map \bar{p}_j' by σ'_j . We start with forms ω_i (of some degrees) on the strata σ_i in $\overline{K(m_1+1, N_i)}$, and with forms ω'_j (of some degrees) on the strata σ'_j in $\overline{K(M_j, n_1+1)}$. Let $\tilde{\Psi}_i = \Psi_i \otimes [\omega_i]_{\sigma_i}$ and $\tilde{\Theta}_j = \Theta_j \otimes [\omega'_j]_{\sigma'_j}$ where Ψ_i 's and Θ_j 's are as above.

We define the composition $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ as

$$\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}} := \frac{\Psi_1 \Psi_2 \dots \Psi_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}} \otimes [\Omega]_\Sigma \quad (22)$$

where the stratum Σ in $\overline{K(m, n)}$ and a form ω on it are defined as follows:

Define the form Ω_1 on stratum σ in $\overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}}$ as $\Omega_1 = \wedge_{i=1}^{n_1+1} \bar{p}_i^* \omega_i$, and define the form Ω_2 on stratum σ' in $\overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}}$ as $\Omega_2 = \wedge_{j=1}^{m_1+1} \bar{p}_j'^* \omega'_j$. Now consider the form $\Omega_1 \boxtimes \Omega_2$ on the stratum $\sigma \times \sigma'$ in $\overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \times \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}}$. According to Conjecture 1(ii), the image of the stratum $\sigma \times \sigma'$ in $\overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \times \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}}$ is a single stratum Σ in $\overline{K(m, n)}$. Define now Σ as the image (it can be not the isomorphic image) of $\sigma \times \sigma'$ with respect to the map $\bar{i}_\sigma: \overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \times \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}} \rightarrow \overline{K(m, n)}$, and define the form Ω on $\overline{K(m, n)}$ as the direct image (the integration along the fiber) of the form $\Omega_1 \boxtimes \Omega_2$ with respect to the restriction of \bar{i}_σ to the stratum $\sigma \times \sigma'$ (which is a trivial bundle). We consider only the case when $\sum_i \text{codim}_{\overline{K(m_1+1, N_i)}} \sigma_i + \sum_j \text{codim}_{\overline{K(M_j, n_1+1)}} \sigma'_j = \text{codim}_{\overline{K_{m_1+1}^{1,\dots,1,n_0,1,\dots,1}} \times \overline{K_{1,\dots,1,m_0,1,\dots,1}^{n_1+1}}} \sigma \times \sigma'$. In this case, the formula above for $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ obeys the L_∞ grading condition. The proof is straightforward.

We define the composition $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ as 0 in all other cases.

The following conjecture has the crucial meaning, at the moment we have no ways to prove it.

Conjecture 2. *The compositions $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}}$ defined above are compatible with the (total) differential in $B^{\bullet\bullet}(m, n)$'s.*

2 The extended Kontsevich space $K(m, n; s)$ and the Propagator

2.1 The definition of the space $K(m, n; s)$

Consider the direct product $P = \mathbb{R}^2 \times \mathbb{R}_+$ of the plane \mathbb{R}^2 with coordinates (x, y) with the half-line \mathbb{R}_+ with the coordinate λ , $\lambda > 0$. We denote by (x, y, λ) the coordinates of a point in $\mathbb{R}^2 \times \mathbb{R}_+$. Consider the disjoint union of P with the two lines: $P_1 = P \sqcup \{(\mathbb{R}, 0, 0)\} \sqcup \{(0, \mathbb{R}, \infty)\}$. Define the following configuration space of points in P_1 :

$$\begin{aligned} \text{Conf}_{m,n;s} = \{ & p_1, \dots, p_m \in (\mathbb{R}, 0, 0), \ p_i \neq p_j \text{ for } i \neq j, \\ & q_1, \dots, q_n \in (0, \mathbb{R}, \infty), \ q_i \neq q_j \text{ for } i \neq j, \\ & t_1, \dots, t_s \in \mathbb{R}^2 \times \mathbb{R}_+, \ t_i \neq t_j \text{ for } i \neq j \} \end{aligned} \quad (23)$$

The action of the 3-dimensional group G^3 on $\text{Conf}_{m,n}$ (see Section 2) can be continued to an action on $\text{Conf}_{m,n;s}$. Indeed, define a product on $\mathbb{R}^2 \times \mathbb{R}_+$ as follows:

$$(x_1, y_1, \lambda_1) \cdot (x, y, \lambda) = (\lambda_1 x + x_1, \lambda_1^{-1} y + y_1, \lambda \lambda_1) \quad (24)$$

This group $\mathbb{R}^2 \times \mathbb{R}$ is exactly G^3 . Then it is clear that it acts on $\text{Conf}_{m,n;s}$ (see formula (8)). This action is free when $m + n + s \geq 3$. Denote in this case the quotient space $\text{Conf}_{m,n;s}/G^3$ by $K(m, n; s)$. It is a smooth manifold of dimension $m + n + 3s - 3$. We are going to compactify this space in a way compatible with the compactification of $K(m, n)$ introduced in Section 2.

2.2 The 3-dimensional Eye and the Propagator

2.2.1 The 3-dimensional Eye

Here we compactify the space $K(0, 0; 2)$ which is 3-dimensional. We call the space $\overline{K(0, 0; 2)}$ the *3-dimensional Eye* by the analogy with the Kontsevich Eye ([K1], Sect. 5.2). The Propagator constructed below will be a form (with singularities) on the 3-dimensional Eye. On the other hand, this simplest example illustrates (the all) new ideas which appear in the compactification of the extended Kontsevich spaces. We describe all strata of codimension 1 in $\overline{K(m, n; s)}$ in the next Subsection.

We consider an oriented pair of points (t_1, t_2) in $\mathbb{R}^2 \times \mathbb{R}_+$. Then using the action of the group G^3 we can fix one of them, say t_1 . Then we suppose that $t_1 = (0, 0, 1)$ and there is no group action. Then we should compactify the space

$$P = \{t = (x, y, \lambda) \in \mathbb{R}^2 \times \mathbb{R}_+, \ (x, y, \lambda) \neq (0, 0, 1)\} \quad (25)$$

It is easy to compactify near the point $(0,0,1)$: we just cut-off a sphere around this point. The other boundary components arise when the point t tends to ∞ . Let us describe them:

First consider the case when the point t tends to infinity when the coordinate λ is finite. Consider the projection $p: (x, y, \lambda) \mapsto (x, y)$. The the two lines (for the values $\lambda = 0$ and $\lambda = \infty$ are the coordinate axis (see Figure 8 below). The degenerations of codimension 1 are then the configurations when *only one* among the two coordinates after the projection tends to infinity. They are "separated" by faces of codimension 2 when the both coordinates x, y tend to ∞ . Let us explain how we compute the dimension of the strata: when one coordinate tends to ∞ , it is defined only up to a finite summand. Therefore the only coordinates on the moduli is another coordinate on the plane (after the projection), and the coordinate λ . Therefore, the strata of codimension 1 are "separated" by the strata of codimension 2 corresponding to the cases when both x, y tend to ∞ . More precisely, they are separated by the lines $x = y$ and $x = -y$ in Figure 8. Thus, we have 4 faces of codimension 1 which are corresponded to the following 4 cases:

- (i) $x \gg 0$, y is finite (positive or negative),
- (ii) $y \gg 0$, x is finite (positive or negative),
- (iii) $x \ll 0$, y is finite (positive or negative),
- (iv) $y \ll 0$, x is finite (positive or negative).

These are the all strata of codimension 1. There are 6 strata of codimension 2. First 4 among are:

- (A) $x \gg 0$, $y \gg 0$,
- (B) $x \ll 0$, $y \gg 0$,
- (C) $x \ll 0$, $y \ll 0$,
- (D) $x \gg 0$, $y \ll 0$.

The last 2 faces of codimension 2 are obtained when λ tends to 0 and to ∞ . We show these strata in Figure 8. Finally, The 3-dimensional Eye is the tetrahedron without a small ball inside (corresponded to the case when t is close to $(0,0,1)$ (see Figure 9). We designated the strata of codimension 1 in Figure 9.

2.2.2 The Propagator

We define a Propagator 2-form as a *closed* 2-form ϕ with *singularities* on the 3-dimensional Eye such that:

- 1) the form ϕ has singularities only at the edge $\{\lambda = 0\}$ of the tetrahedron,

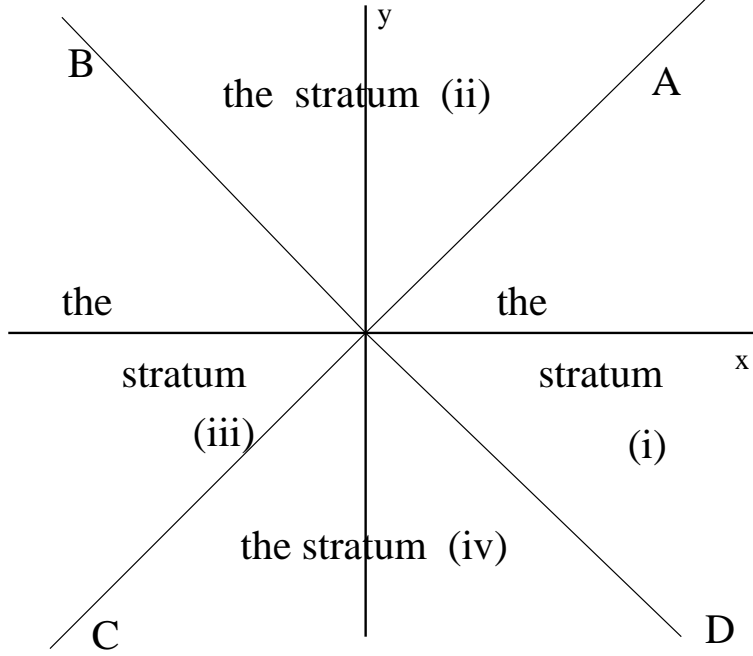


Figure 4: The strata of codimension 1 in 2-dimensional projection

- 2) the restriction of the form ϕ to the 2-dimensional sphere (around the point $(0, 0, 1)$) is the volume form on the sphere normalized such that the integral over the sphere is equal to 1,
- 3) the restriction of the form ϕ to the other boundary components (besides the sphere and the interval $\{\lambda = 0\}$) is 0,
- 4) the singularity at the edge $\{\lambda = 0\}$ has the form described below.

Consider the rectangle P_λ which is the horizontal section of the tetrahedron by the plane $z = \lambda$ where λ is close to 0 (see Figure 9). Let x and y be the coordinates on the rectangle. We say that $x \in [-1, 1]$ and $y \in [-\epsilon, \epsilon]$ for a "very small" ϵ . Roughly speaking, we want that in the limit $\lambda \rightarrow 0$ the restriction of the Propagator 2-form to our rectangle would be $f(x)\delta(y)dx \wedge dy$ where $\delta(y)$ is the Dirac delta-function with the support at $y = 0$, and $f(x)$ is any *positive* function with the support in the open interval $(-1, 1)$ and such that $\int_{-1}^1 f(x)dx = 1$. Notice that the Propagator 2-form should be not defined when $t \in \{\lambda = 0\}$, there we have the Propagator 1-form which is, by definition, the 1-form $f(x)dx$ where $f(x)$ is as above. So, the above description is *not* a description at $\lambda = 0$, but when $\lambda \rightarrow 0$.

Let us prove that such a Propagator 2-form exists (it is more or less clear that if it exists it is defined up to a homotopy in an appropriate sense):

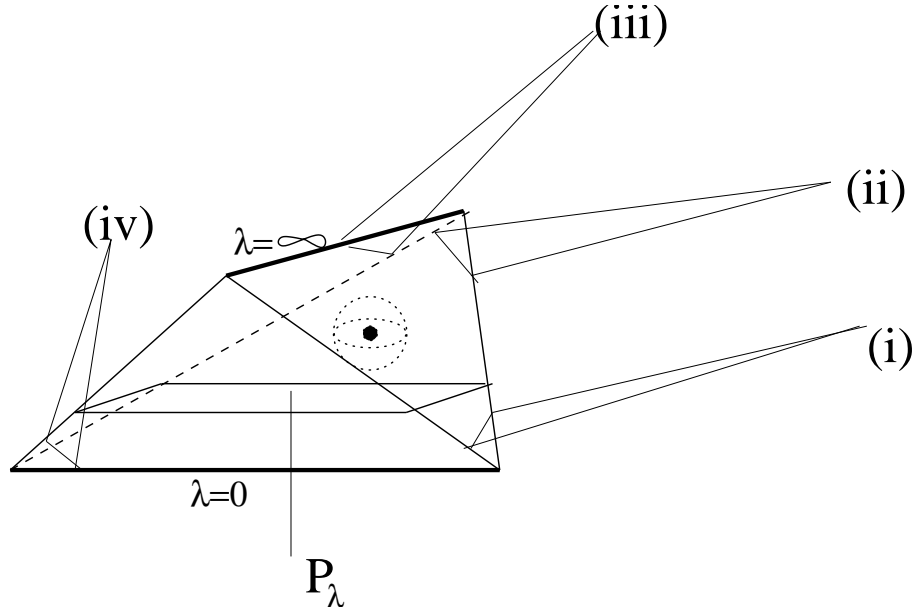


Figure 5: The 3-dimensional Eye

Consider the part of the 3-dimensional Eye which is above the rectangle P_λ (see Figure 9), where λ is a sufficiently small number (it is important only that the cut-off sphere is above it). We call this part the *truncated* 3-dimensional Eye.

Definition. We say that a map of the truncated 3-dimensional Eye to the 2-dimensional sphere S^2 is *spherical* if it maps the sphere inside the tetrahedron homototically to the sphere S^2 , maps all the boundary of the truncated tetrahedron *except* the rectangle P_λ to a point $p \in S^2$, and it maps the domain $D \subset P_\lambda$ to the sphere S^2 and $P_\lambda \setminus D$ to the point $p \in S^2$ such that the factor-space $D/\partial D$ maps isomorphically to S^2 in the way compatible with the orientation. Here the domain D is any domain like it is shown in Figure 10. It should be any simply-connected domain having a non-empty intersection with $\{y = 0\} \subset P_\lambda$.

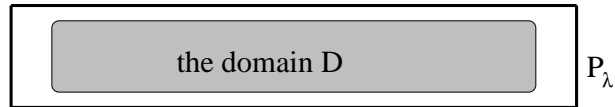


Figure 6: The domain D in the rectangle P_λ

It is clear that such a map exists and is homotopically unique. Then, starting with a spherical map, we can easily construct a Propagator 2-form. First, define it on the

truncated rectangle as $\pi^*(\omega)$ where π is a spherical map, and ω is the volume form on S^2 normalized such that $\int_{S^2} \omega = 1$. Then consider the cone which is the complement to the truncated tetrahedron in the tetrahedron, and continue the Propagator form to a form with singularities in a natural way. We should be careful about the smoothness of the form, but it is possible to make it smooth. Then we define the Propagator 1-form at the edge $\{\lambda = 0\}$ as follows: we consider the direct image of the restriction of the Propagator 2-form on P_λ with respect to the projection of P_λ to the $P_\lambda \cap \{y = 0\}$.

2.3 The compactification $\overline{K(m, n; s)}$

Here we define a compactification $\overline{K(m, n; s)}$ of the space $K(m, n; s)$.

Recall our notations $\Sigma_{1,3}$, $\Sigma_{2,2}$, and $\Sigma_{3,1}$ for the possible samples of 4 points on the 2 boundary lines (see Section 1.1). Now denote by Σ^2 the all possible pairs of points in $K(m, n; s)$ among the s inner points, and denote by Σ_1^1 the all possible pairs of points in $K(m, n; s)$ one of which is an inner point, and another is a points on the upper or lower boundary line. For $\sigma \in \Sigma^2$ we have a map $r_\sigma: K(m, n; s) \rightarrow K(0, 0; 2)$, and as well for $\sigma \in \Sigma_1^1$ we have a map $r_\sigma: K(m, n; s) \rightarrow K(1, 0; 1)$ or to $K(0, 1; 1)$. In the sequel we will denote the last two spaces by one symbol $K(\P; 1)$. We have the direct product of the maps r_σ :

$$r = \prod_{\sigma \in \Sigma_{1,3} \sqcup \Sigma_{3,1} \sqcup \Sigma_{2,2} \sqcup \Sigma^2 \sqcup \Sigma_1^1} r_\sigma: K(m, n; s) \rightarrow \prod_{\sigma \in \Sigma_{1,3}} K(1, 3) \times \prod_{\sigma \in \Sigma_{2,2}} K(2, 2) \times \prod_{\sigma \in \Sigma_{3,1}} K(3, 1) \times \prod_{\sigma \in \Sigma^2} K(0, 0; 2) \times \prod_{\sigma \in \Sigma_1^1} K(\P; 1) \quad (26)$$

It is clear that the map r is an imbedding. We know how to compactify each space in the right-hand side. It is the 3-dimensional eye for $K(0, 0; 2)$, and a part of the 3-dimensional eye (which we imbed into the 3-dimensional eye) for $K(\P; 1)$. Then we can imbed each space in the r.h.s to its compactifacation, and then take the closure of the image in this imbedding. This is, by definition, our compactification $\overline{K(m, n; s)}$.

Remark. In the Kontsevich paper on deformation quantization, one also can compactify the space $C_{m,n}$ in this way. But for a pair of inner points we should consider the corresponding point of the Kontsevich eye. Kontsevich attached just the corresponding angle for a pair of inner points. In this way, he does not get an imbedding, and therefore automatically he gets a wrong compactifacation, which does not coincide with his right compactification described in the terms of trees. When one considers a point of the Kontsevich eye instead of the corresponding angle, we get the right described in trees compactification.

We claim the following lemma:

Lemma. (i) The space $\overline{K(m, n; s)}$ is naturally a manifold with corners,

(ii) The projection $p: K(m, n; s) \rightarrow K(m, n)$ can be uniquely continued to a map $\bar{p}: \overline{K(m, n; s)} \rightarrow \overline{K(m, n)}$ which is a map of manifolds with corners.

□

Now we describe all strata of codimension 1 in $\overline{K(m, n; s)}$. We define the dimension "of a point" as follows: it is equal to k if we need to know k_1 1-dimensional 4-point ratios, k_2 3-dimensional 4-point ratios, $k_1 + 3k_2 = k$, and this number can be not made less.

The images with respect to \bar{p} of the strata of codimension 1 in $\overline{K(m, n; s)}$ are either the stratum of codimension 0 in $\overline{K(m, n)}$ (we call these strata of codimension 1 in $\overline{K(m, n; s)}$ *the strata of codimension 1 of the first type*), or are strata of codimension 1 in $\overline{K(m, n)}$ (we call them *the strata of codimension 1 of the second type*). First list all the strata of codimension 1 in $\overline{K(m, n; s)}$.

Strata of codimension 1 of the first type

S1.1 s_1 points from the s "inner" points move close to each other, to a finite point, and for a finite value of λ , $s_1 \geq 2$,

S1.2 s_1 "inner" points at a finite distance from each other move to infinity for a finite λ , and such that only one coordinate among x, y tends to ∞ .

Strata of codimension 1 of the second type

S2 A typical stratum is drawn in Figure 7. The reader should remember that the two lines are *crossing* (that means two lines in a 3-dimensional space which do not intersect). On these two boundary lines we have exactly the picture from Figure 2. The points inside the circle **A** are in finite distances everywhere. The points inside the circles **B** and **C** are infinitely far to the right and to the left from the circle **A**. When we apply an infinite shift to the circle **B** the points inside it will be everywhere in the finite distances, the same is true for the circle **C**. (Remember that the boundary lines are *crossing*!). On the lower line we have several groups of points infinitely close to each other, the distances between the groups are finite. The lower and the upper infinities have the same order. When we apply the transform $(0, 0, \lambda) \in G^3$ for infinite λ to Figure 7, the lower line will look like the upper, and vice versa.

Remark. It was a remarkable insight of Maxim Kontsevich that the lines should be crossing.

2.4 The direct image $\bar{p}_*(\omega)$ of a closed form ω on $\overline{K(m, n; s)}$

Consider the projection $\bar{p}: \overline{K(m, n; s)} \rightarrow \overline{K(m, n)}$. Let ω be a smooth differential form (not necessarily closed) on $\overline{K(m, n; s)}$. The smoothness here means that it is a continuous form such that the restriction of it to each open stratum is smooth. What is the

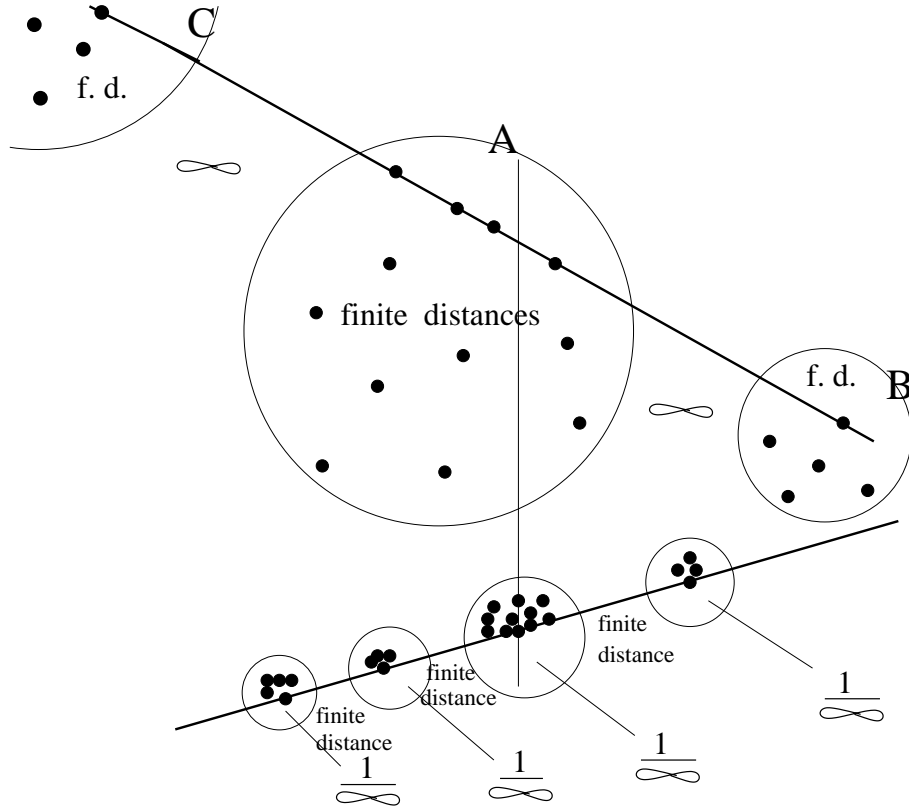


Figure 7: A typical stratum S2 of codimension 1

"right definition" of the pushforward $\bar{p}_*(\omega)$ which *should be an element of the bicomplex* $B^{\bullet\bullet}(m, n)$? The usual definition as the integration over the fiber is not good because it is a trivial bundle over a single stratum σ , but the fiber changes from stratum to stratum.

Definition. Let $\bar{p}: M_1 \rightarrow M_2$ be a map of manifolds with corners. Let ω be a homogeneous element of some degree $k - \dim M_1$ in $B^{\bullet\bullet}(M_1)$. It means that $\omega = \omega_0^i + \omega_1^j + \omega_2^t + \dots$ where ω_0^i is a form of degree k on the i th stratum of codimension 0, ω_1^j is a form of degree $k - 1$ on the j th stratum of codimension 1, and so on. Define the direct image $\bar{p}_*: B^{\bullet\bullet}(M_1) \rightarrow B^{\bullet\bullet}(M_2)$ as the following map of degree 0: The image $\bar{p}_*(\omega)$ is defined as the sum $\sigma_{m,n} \bar{p}_* \omega_m^n$ over all m, n . Here by \bar{p}_* we mean the integration over the fiber. By the definition of a map of manifold with corners, the image of each stratum σ in M_1 is a stratum $\bar{p}_*(\sigma)$ in M_2 , and $\dim F + \dim \bar{p}_*(\sigma) = \dim \sigma$ where F is the fiber over $\bar{p}_*(\sigma)$. The degree of the form $\bar{p}_*(\omega_m^n)$ is $\deg \omega_m^n - \dim F$, therefore $\deg \omega_m^n - \dim \sigma = (\deg \omega_m^n - \dim F) - \dim \bar{p}_*(\sigma) = \deg \bar{p}_*(\omega_m^n) - \dim \bar{p}_*(\sigma)$. Therefore, we obtain a map $\bar{p}_*: B^{\bullet\bullet}(M_1) \rightarrow B^{\bullet\bullet}(M_2)$ of degree 0.

As we will see now, the following result is just an application of the Stokes formula.

Proposition. *The map $\bar{p}_*: B^{\bullet\bullet}(M_1) \rightarrow B^{\bullet\bullet}(M_2)$ is a map of (total) complexes.*

Proof. First of all, we need the following "relative" version of the Stokes formula:

Lemma. *Let E and B be smooth manifolds with boundary, and let $p: E \rightarrow B$ be the trivial bundle with fiber F which is a compact manifold with boundary (that is, $E = B \times F$). Let ω be a differential form on E , we denote by $p_*\omega$ the direct image which is the integral of ω along the fiber F . Then we have:*

$$p_*d\omega = (p|_{\partial F})_*\omega + dp_*\omega \quad (27)$$

where d is the de Rham differential.

Proof. As p is a trivial bundle, we can decompose the de Rham differential on E into the sum of its horizontal component d_{hor} , and its vertical component d_{vert} , $d = d_{hor} + d_{vert}$. Therefore, $p_*d\omega = \int_F d_{hor}\omega + \int_F d_{vert}\omega$. By the Stokes formula, the second summand is $\int_{\partial F} \omega$. The first summand is $d_{hor} \int_F \omega = dp_*\omega$. \square

Now we prove the Proposition. It is enough to consider the case when the element ω of the bicomplex has only one component, which is a form Ω on a simplex σ . We need to prove:

$$\bar{p}_*(d\Omega) - \bar{p}_*\Omega|_{\partial\sigma} = d\bar{p}_*\Omega - (\bar{p}_*\Omega)|_{\partial\bar{p}\sigma} \quad (28)$$

which follows immediately from Lemma above and from the straightforward relation

$$\bar{p}_*\Omega|_{\partial\sigma} - (\bar{p}_*\Omega)|_{\bar{p}\partial\sigma} = (\bar{p}|_{\partial F})_*\Omega \quad (29)$$

where F is the fiber over σ . \square

3 The formality equation and the quantization of Lie bialgebras

3.1 The first look at the formality equation

The components \mathcal{U}_k of the analog of Kontsevich L_∞ morphism is a sum over graphs. To each graph we attach two things: these are a map from $\text{Hom}(A^{\otimes m}, A^{\otimes n})$, and a top degree differential form on a configuration space. Moreover, this top degree differential form is the product of the differential forms attached to the edges of the graph (we do not assume that these are necessarily 1-forms). Let us try to outline the situation, starting with these very general principles.

We want to construct a map

$$\mathcal{U}_k: \wedge^k(\mathfrak{g}_1) \rightarrow \mathfrak{g}_2[1-k] \quad (30)$$

where as above $\mathfrak{g}_1 = \oplus_{m,n \geq 1} \text{Hom}(\wedge^m(V), \wedge^n(V))[-m - n + 2]$, and $\mathfrak{g}_2 = \oplus_{m,n \geq 1} \text{Hom}(A^{\otimes m}, A^{\otimes n})[-m - n + 2]$. We know that \mathfrak{g}_1 is the Poisson Lie algebra, and expect for an L_∞ -structure on \mathfrak{g}_2 for which U is an L_∞ map. This explains the shift of degree on $1 - k$ in (30).

Let us compute the degrees. We compute $\mathcal{U}_k(\gamma_1, \dots, \gamma_k)$. Suppose that $\gamma_i \in \text{Hom}(\wedge^{a_i} V, \wedge^{b_i} V)$. Suppose $\mathcal{U}_k(\gamma_1, \dots, \gamma_k) \in \text{Hom}(A^{\otimes m}, A^{\otimes n})$. Then it follows from (30) that

$$\sum_{i=1}^k (a_i + b_i - 2) + (1 - k) = m + n - 2 \quad (31)$$

The last equation is equivalent to

$$\sum_{i=1}^k a_i + \sum_{i=1}^k b_i = m + n + 3k - 3 \quad (32)$$

Now we can express the left hand side through the number of edges of the graph. Namely, we distinguish the "inner" edges and the "external" edges. We suppose that we have some boundary in the configuration space with $m + n$ points there, and the interior, with k points. There are no edges between $m + n$ boundary points. Then each edge contain *at least* one inner vertex. We call the edge internal, if the both ends of it are inner, and inner, if only one among the two ends is inner. Then it is clear that

$$2\sharp E_{inner} + \sharp E_{external} = \sum_{i=1}^k a_i + \sum_{i=1}^k b_i \quad (33)$$

(Indeed, the number of edges starting and ending at the i th inner point is $a_i + b_i$ by the assumption, we count then each inner edge twice and each external edge one time). Finally, we have:

$$2\sharp E_{inner} + \sharp E_{external} = m + n + 3k - 3 \quad (34)$$

We interpret the last equation as follows: the dimension of the configuration space should be equal to $3k + m + n - 3$, and we should attach a 2-form to each inner edge, and a 1-form to each external edge. The dimension $3k + m + n - 3$ of the configuration space means that the "boundary" points should belong to some 1-dimensional space, inner points—to a 3-dimensional space, and there is an action of a 3-dimensional group on the configuration space.

We made this computation in the very beginning of this work together with Maxim Kontsevich, and then he invented the spaces $K(m, n; s)$ as an appropriate candidates.

3.2 The admissible graphs and the corresponding operators in $\text{Hom}(A^{\otimes m}, A^{\otimes n})$

3.2.1 The admissible graphs

We will integrate over the spaces $K(m, n; s)$ some differential forms of the top degree, associated with *admissible graphs*. We associate with any *inner* edge of this graph the Propagator 2-form constructed above, and with any *external* edge the corresponding 1-form.

Definition. Admissible graph Γ is an oriented graph with labels such that:

- 1) the set of vertices V_Γ is $\{1, \dots, s\} \sqcup \{\underline{1}, \dots, \underline{m}\} \sqcup \{\bar{1}, \dots, \bar{n}\}$, $3s + m + n \geq 3$, the vertices from the set $\{1, \dots, s\}$ are called vertices of the first type, the vertices from the set $\{\underline{1}, \dots, \underline{m}\}$ are called the lower vertices of the second type, and the vertices from the set $\{\bar{1}, \dots, \bar{n}\}$ are called the upper vertices of the second type,
- 2) every edge $(v_1, v_2) \in E_\Gamma$ starts at a vertex of the first type or at an upper vertex of the second type, and ends at a vertex of the first type or at a lower vertex of second type, $v_1 \in \{1, \dots, s\} \sqcup \{\bar{1}, \dots, \bar{n}\}$, $v_2 \in \{1, \dots, s\} \sqcup \{\underline{1}, \dots, \underline{m}\}$, if both $v_1, v_2 \in \{1, \dots, s\}$, the edge is called inner, other edges are called external, there are no (external) edges between two vertices of the second type,
- 3) there are no simple loops, that is edges of the type (v, v) , there are no multiple external edges, but there are can be multiple inner edges,
- 4) for every vertex k of the first type the sets of edges

$$\text{Star}(k) = \{(v_1, v_2) \in E_\Gamma | v_1 = k\}$$

and

$$\text{End}(k) = \{(v_1, v_2) \in E_\Gamma | v_2 = k\}$$

are labeled by symbols $(s_k^1, \dots, s_k^{\#\text{Star}(k)})$ and $(e_k^1, \dots, e_k^{\#\text{End}(k)})$.

The simplest examples of admissible graphs are shown in Figure 8.

Remark. Notice that we do *not* fix the number of the edges of Γ in this definition.

3.2.2 The polydifferential operators associated with admissible graphs

For an admissible graph Γ with s vertices of first type, (m, n) vertices of second type, we associate a map

$$\Phi_\Gamma: \otimes^s \Lambda^\bullet(V \oplus V^*) \rightarrow \text{Hom}(A^{\otimes m} \rightarrow A^{\otimes n})[1 - s]$$

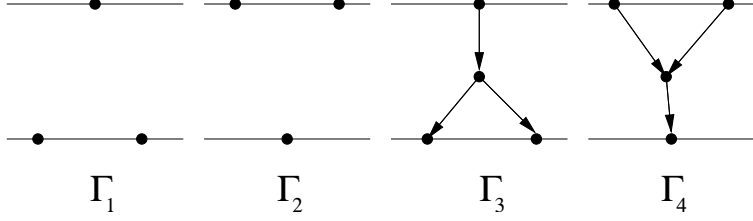


Figure 8: The simplest admissible graphs

Recall that V is a Lie bialgebra, $A = S^\bullet(V^*)$ is a free commutative cocommutative associative bialgebra, and

$$\deg(\text{Hom}(\wedge^i(V), \wedge^j(V))) = \deg(\text{Hom}(A^{\otimes i}, A^{\otimes j})) = i + j - 2$$

Let $\gamma_1, \dots, \gamma_s \in \text{Hom}(\Lambda^\bullet(V), \Lambda^\bullet(V))$. Then $\Phi_\Gamma(\gamma_1, \dots, \gamma_s)$ is non-zero only if $\gamma_i \in \text{Hom}(\wedge^{\sharp \text{Star}(i)} V, \wedge^{\sharp \text{End}(i)} V)$.

We are going to write a formula for

$$\Phi_\Gamma(\gamma_1, \dots, \gamma_s)(f_1, \dots, f_m) \in A^{\otimes n}$$

The formula is the sum over all labelings of the edges of Γ by indices running from 1 to d , $d = \dim V$:

$$\Phi_\Gamma = \sum_{I: E_\Gamma \rightarrow \{1, \dots, d\}} \Phi_{\Gamma, I} \quad (35)$$

where each $\Phi_{\Gamma, I}$ is

$$\Phi_{\Gamma, I} = \Delta^n \left(\prod_{\substack{v \in V_\Gamma, \\ v \neq \vec{i} \text{ for some } i}} \Psi_v \right) \cdot (\otimes_{i=1}^n \Psi_{\vec{i}}) \quad (36)$$

The product \cdot here means the product

$$(a_1 \otimes \dots \otimes a_n) \cdot (b_1 \otimes \dots \otimes b_n) = ((a_1 \cdot b_1) \otimes \dots \otimes (a_n \cdot b_n)).$$

At each vertex v of the first type, the function Ψ_v is a constant: it is the matrix element for γ_v ,

$$\Psi_v = \langle \gamma_v(x^{I(s_v^1)} \wedge \dots \wedge x^{I(s_v^{\sharp \text{Star} v})}), x^{I(e_v^1)} \wedge \dots \wedge x^{I(e_v^{\sharp \text{End} v})} \rangle \quad (37)$$

Now at each *lower* vertex of the second type, the function Ψ_v is a partial derivative of f_v :

$$\Psi_v = \left(\prod_{e \in E_\Gamma, e = (*, v)} \partial_{I(e)} \right) f_v \quad (38)$$

and the function associated with each vertex v of *upper* first type is

$$\Psi_v = \left(\prod_{e \in E_\Gamma, e=(v,*)} x^{I(e)} \right) \quad (39)$$

Now the formula for the summand $\Phi_{\Gamma,I}$ is given by formula (36).

The formula (36) is equivariant with respect to the linear group $GL(V)$ because it uses only invariant operations.

Example. For the first graph Γ_1 drawn in Figure 8, the map $\Phi_{\Gamma_1}: A^{\otimes 2} \rightarrow A$ is the product:

$$\Phi_{\Gamma_1}(f_1 \otimes f_2) = f_1 \cdot f_2$$

For the second graph Γ_2 in Figure 8, the map $\Phi_{\Gamma_2}: A \rightarrow A^{\otimes 2}$ is the coproduct:

$$\Phi_{\Gamma_2}(f) = \Delta(f)$$

Consider the third graph Γ_3 in Figure 8. It defines a map $\Phi_{\Gamma_3}: A^{\otimes 2} \rightarrow A$ when in the unique inner vertex is placed the structure map $c_{ij}^k: \wedge^2 V \rightarrow V$. By the definition above,

$$\Phi_{\Gamma_3}(f_1 \otimes f_2) = \sum_{i,j,k=1}^d c_{ij}^k x_k \partial_i(f_1) \partial_j(f_2) = \{f_1, f_2\}$$

where $\{, \}$ here stands for the Kostant-Kirillov Poisson bracket on V^* .

For the fourth graph Γ_4 drawn in Figure 8, we have a map $\Phi_{\Gamma_4}: A \rightarrow A^{\otimes 2}$ where in the unique inner vertex is placed the cobracket $d_i^{jk}: V \rightarrow \wedge^2 V$. By our definition,

$$\Phi_{\Gamma_4}(f) = \sum_{i,j,k=1}^d d_i^{jk} \Delta(\partial_i f) \cdot (x_j \otimes x_k)$$

which is equal to the Poisson cobracket of f .

3.3 The differential form Ω_Γ associated with an admissible graph Γ

Consider an admissible graph Γ with m lower vertices of the first type, n upper vertices of the first type, and s inner vertices (of the second type). Suppose it has $\sharp E_{inner}$ edges connecting the inner vertices and $\sharp E_{external}$ edges one the end-points of which is a vertex of the first type (upper or inner). We attach to the ℓ th inner vertex of Γ an element γ_ℓ of the Poisson Lie algebra $\mathfrak{g}_1^\bullet = \oplus_{m,n \geq 1} \text{Hom}(\wedge^m(V), \wedge^n(V))[-m-n+2]$ where V is a finite-dimensional vector space. If $\gamma_\ell \in \text{Hom}(\wedge^{m_\ell}(V), \wedge^{n_\ell}(V))$ than at the inner vertex ℓ there are n_ℓ edges starting at this vertex, and m_ℓ edges ending at this vertex. Then we have:

$$\sum_{\ell=1}^s \deg \gamma_\ell = 2\sharp E_{inner} + \sharp E_{external} - 2s \quad (40)$$

Now we attach to the graph a form Ω_Γ on the space $\overline{K(m, n; s)}$ as follows:

Each edge (inner or external) α of Γ defines a map $j_\alpha: \overline{K(m, n; s)} \rightarrow \overline{K(0, 0; 2)}$ to the 3-dimensional Eye (see Section 2.2). We have constructed the Propagator 2-form ϕ (for inner edges), and the Propagator 1-form ϕ_0 (for external edges). Now define

$$\Omega_\Gamma = \bigwedge_{\alpha \in E_{inner}} j_\alpha^*(\phi) \wedge \bigwedge_{\alpha \in E_{external}} j_\alpha^*(\phi_0) \quad (41)$$

which is a form (with singularities) of degree $2\sharp E_{inner} + \sharp E_{external}$ on the space $\overline{K(m, n; s)}$. (The labeling of Γ allows to fix the ordering in the wedge product). The singularities are only δ -singularities, and the form absolutely converges.

Remark. More precisely, we consider an object like the product of δ -functions which does not rigorously exist. For this we define all constructions above with the Propagator and the configuration spaces defined from the *truncated* 3-dimensional Eye (see Section 2.2.2) and then take the limit when the rectangle P_λ tends to the interval ($\lambda = 0$). This is a kind of regularization we use here. In the sequel we always have in mind this regularization and never consider these problems.

In Section 3.2.2 we attached also an operator $\Phi_\Gamma(\gamma_1, \dots, \gamma_s) \in \text{Hom}(A^{\otimes m}, A^{\otimes n})$ where $A = S(V^*)$.

Define now a map

$$\mathcal{U}_\Gamma: \wedge^s(\mathfrak{g}_1^\bullet) \rightarrow \widetilde{K}^\bullet_{GS}[1-s] \quad (42)$$

defined as

$$\mathcal{U}_\Gamma(\gamma_1 \wedge \dots \wedge \gamma_s) = \frac{1}{s!} \text{Alt}_\gamma \Phi_\Gamma(\gamma_1, \dots, \gamma_s) \otimes \overline{p}_* \Omega_\Gamma \quad (43)$$

Here $\overline{p}: \overline{K(m, n; s)} \rightarrow \overline{K(m, n)}$ is the natural projection, and $\overline{p}_*(\Omega_\Gamma)$ is considered as an element of $B^{\bullet\bullet}(m, n)$ which has the only one nonzero component—the form $\overline{p}_*(\Omega_\Gamma)$ on the top dimensional open stratum in $\overline{K(m, n)}$.

Let us prove that the shift of grading is correct: indeed, by (40), $\sum_\ell \deg \gamma_\ell = 2\sharp E_{inner} + \sharp E_{external} - 2s$. One needs to prove that $2\sharp E_{inner} + \sharp E_{external} - 2s + (1-s) = \deg \Phi_\Gamma + (\deg \overline{p}_*(\Omega_\Gamma) - (m+n-3))$. But $\deg \Phi_\Gamma = m+n-2$ and $\deg \overline{p}_*(\Omega_\Gamma) = 2\sharp E_{inner} + \sharp E_{external} - 3s$. We are done.

3.4 The "formality" equation: a Conjecture and a Theorem

Fix $s \geq 0$. Define

$$\mathcal{U}_s(\gamma_1, \dots, \gamma_s) = \sum_{m, n, m+n \geq 3} \sum_{\Gamma \in \Gamma_{m, n; s}} \mathcal{U}_\Gamma(\gamma_1, \dots, \gamma_s) \quad (44)$$

where $\Gamma_{m, n; s}$ are the graphs Γ with m lower vertices of the first type, n upper vertices of the first type, and s inner vertices. When $\gamma_1, \dots, \gamma_s$ are fixed, the number $2\sharp E_{inner} +$

$\sharp E_{external}$ is fixed, and therefore the degrees $\deg \Omega_\Gamma$ and $\deg \bar{p}_*(\Omega_\Gamma)$ are fixed. Consider such m, n that $\deg \bar{p}_* \Omega_\Gamma = m + n - 4$ (for 1 less than the maximal).

The form $\bar{p}_* \Omega_\Gamma$ is not closed: by Lemma 2.4,

$$d\bar{p}_* \Omega_\Gamma = \bar{p}_*(d\Omega_\Gamma) - \bar{p}|_{\partial F} \Omega_\Gamma \quad (45)$$

The first summand is 0 because the Propagator is closed. Finally, we have:

$$d\bar{p}_* \Omega_\Gamma + \bar{p}|_{\partial F} \Omega_\Gamma = 0 \quad (46)$$

Both forms are top degree forms on $\overline{K(m, n)}$. Let us integrate the equation (46) over the fundamental cycle in $\overline{K(m, n)}$. We have:

$$\int_{\overline{K(m, n)}} d\bar{p}_* \Omega_\Gamma + \int_{\overline{K(m, n)}} \bar{p}|_{\partial F} \Omega_\Gamma = 0 \quad (47)$$

Consider the equation:

$$\sum_{\Gamma \in \Gamma_{m, n; s}^{m+n+3s-4}} \Phi_\Gamma(\gamma_1, \dots, \gamma_s) \cdot \left(\int_{\overline{K(m, n)}} d\bar{p}_* \Omega_\Gamma + \int_{\overline{K(m, n)}} \bar{p}|_{\partial F} \Omega_\Gamma \right) = 0 \quad (48)$$

where $\Gamma_{m, n; s}^{m+n+3s-4}$ denotes the graphs with m lower vertices of first type, n upper vertices of first type, s vertices of second type, and for which $2\sharp E_{inner} + \sharp E_{external} = m + n + 3s - 4$, and F is the fiber.

By the usual Stokes formula, we can rewrite:

$$\int_{\overline{K(m, n)}} d\bar{p}_* \Omega_\Gamma = \int_{\partial \overline{K(m, n)}} \bar{p}_* \Omega_\Gamma \quad (49)$$

It is clear that only boundary strata of codimension 1 will contribute to these formulas. These strata have been described above. Now we prove the following theorem:

Theorem. For fixed $s \geq 1$ we have the following relation for each m, n :

$$\begin{aligned} & \sum_{\text{codim } \sigma=1} \int_{\sigma} N_{\{w_i\}} \sum_{\{i_k\} \in \Sigma_s} \sum_{\{w_j\}} \pm \left(\frac{\mathcal{U}_{w_1}(\gamma_{i_1}, \dots, \gamma_{i_{w_1}}) \dots \mathcal{U}_{w_{\ell}-w_{\ell-1}}(\gamma_{i_{w_{\ell-1}}+1}, \dots, \gamma_{i_{w_{\ell}}})}{\mathcal{U}_{w_{\ell+1}-w_{\ell}}(\gamma_{i_{w_{\ell}+1}}, \dots, \gamma_{i_{w_{\ell+1}}}) \dots \mathcal{U}_{s-w_{\ell+\ell'-1}}(\gamma_{i_{w_{\ell+\ell'-1}+1}}, \dots, \gamma_{i_s})} \right)_{\sigma} \\ & + \int_{\overline{K(m,n)}} \sum_{1 \leq i < j \leq s} \pm \mathcal{U}_{s-1}(\{\gamma_i, \gamma_j\}, \gamma_1, \dots, \hat{\gamma}_i, \dots, \hat{\gamma}_j, \dots, \gamma_s) = 0 \end{aligned} \quad (50)$$

This is our "formality" theorem. Let us explain what is written here. We take the sum over all strata σ of codimension 1 in $\overline{K(m,n)}$. We associate with each stratum of codimension 1 the corresponding operation $\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}}$ where all $\tilde{\Psi}$'s and $\tilde{\Theta}$'s are our components \mathcal{U}_{α} . We write this operation now as $\left(\frac{\mathcal{U}_{w_1}(\gamma_{i_1}, \dots, \gamma_{i_{w_1}}) \dots \mathcal{U}_{w_{\ell}-w_{\ell-1}}(\gamma_{i_{w_{\ell-1}}+1}, \dots, \gamma_{i_{w_{\ell}}})}{\mathcal{U}_{w_{\ell+1}-w_{\ell}}(\gamma_{i_{w_{\ell}+1}}, \dots, \gamma_{i_{w_{\ell+1}}}) \dots \mathcal{U}_{s-w_{\ell+\ell'-1}}(\gamma_{i_{w_{\ell+\ell'-1}+1}}, \dots, \gamma_{i_s})} \right)_{\sigma}$. The numbers ℓ and ℓ' are uniquely defined from the combinatorics of σ . The numbers $\{w_i\}$ are not defined uniquely, we take the summation over all possibilities. Finally, we alternate over γ_i 's, taking the sum over all permutations from the permutation group Σ_s . *The idea is that in the components \mathcal{U}_i the corresponding factor in $B^{\bullet\bullet}$ could be not the top degree form, we take the operation on them, and in the result (when we integrate) only the top degree component will contribute* (see also the definition of the Integral map in Section 3.4.4). *Therefore, our condition on s, m, n and the graphs Γ is that $\Gamma \in \Gamma_{m,n;s}^{m+n+3s-4}$. It means that in (50) we take the sum over all such graphs.* The combinatorial factor $N_{\{w_i\}}$ is $\frac{1}{\prod_{i=1}^{\ell+\ell'} w_i!}$.

We prove the Theorem in Sections 3.4.2-3.4.3 below. First of all, discuss that the Gerstenhaber-Schack differential [GS] is hidden in the first summand of the equation (50).

3.4.1 The "formality" relation and the Gerstenhaber-Schack differential

First of all, recall what the Gerstenhaber-Schack differential of a (co)associative bialgebra A is. Recall, that it is a differential on the Gerstenhaber-Schack complex

$$K_{GS}(A) = \oplus_{m,n \geq 1} \text{Hom}(A^{\otimes m}, A^{\otimes n})[-m-n+2] \quad (51)$$

Now let $\Psi: A^{\otimes m} \rightarrow A^{\otimes n} \in K_{GS}^{m+n-2}(A)$. We are going to define the Gerstenhaber-Schack differential $d_{GS}(\Psi) \in \text{Hom}(A^{\otimes(m+1)}, A^{\otimes n}) \oplus \text{Hom}(A^{\otimes m}, A^{\otimes(n+1)})$. Denote the projections of d_{GS} to the first summand by $(d_{GS})_1$, and the projection to the second summand by

$(d_{GS})_2$. The formulas for $(d_{GS})_1$ and $(d_{GS})_2$ are:

$$\begin{aligned}
& (d_{GS})_1(\Psi)(a_0 \otimes \cdots \otimes a_m) = \\
& \Delta^n(a_0) * \Psi(a_1 \otimes \cdots \otimes a_m) \\
& + \sum_{i=0}^{m-1} (-1)^{i+1} \Psi(a_0 \otimes \cdots \otimes (a_i * a_{i+1}) \otimes \cdots \otimes a_m) \\
& (-1)^{m-1} \Psi(a_0 \otimes \cdots \otimes a_{m-1}) * \Delta^n(a_m)
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
& (d_{GS})_2(\Psi)(a_1 \otimes \cdots \otimes a_m) = \\
& (\Delta^{(1)}(a_1) * \Delta^{(1)}(a_2) * \cdots * \Delta^{(1)}(a_m)) \otimes \Psi(\Delta^{(2)}(a_1) \otimes \cdots \otimes \Delta^{(2)}(a_m)) \\
& + \sum_{i=1}^n (-1)^i \Delta_i \Psi(a_1 \otimes \cdots \otimes a_m) \\
& + (-1)^{n+1} \Psi(\Delta^{(1)}(a_1) \otimes \Delta^{(1)}(a_2) \otimes \cdots \otimes \Delta^{(1)}(a_m)) \otimes (\Delta^{(2)}(a_1) * \Delta^{(2)}(a_2) * \cdots * \Delta^{(2)}(a_m))
\end{aligned} \tag{53}$$

Here we symbolically write $\Delta(a) = \Delta^{(1)}(a) \otimes \Delta^{(2)}(a)$ having in mind the sum of several such terms, $\Delta(a) = \sum_i \Delta_i^{(1)}(a) \otimes \Delta_i^{(2)}(a)$, where Δ is the coproduct in A , $*$ is the product in A , and $\Delta_i = \text{Id} \otimes \cdots \otimes \text{Id} \otimes \Delta \otimes \text{Id} \otimes \cdots \otimes \text{Id}$ where Δ is applied to the i -th factor.

Now we show that the all terms in formulas (52) and (53) are corresponded to some boundary strata of codimension 1 in $\overline{K}(m+1, n)$ or $\overline{K}(m, n+1)$.

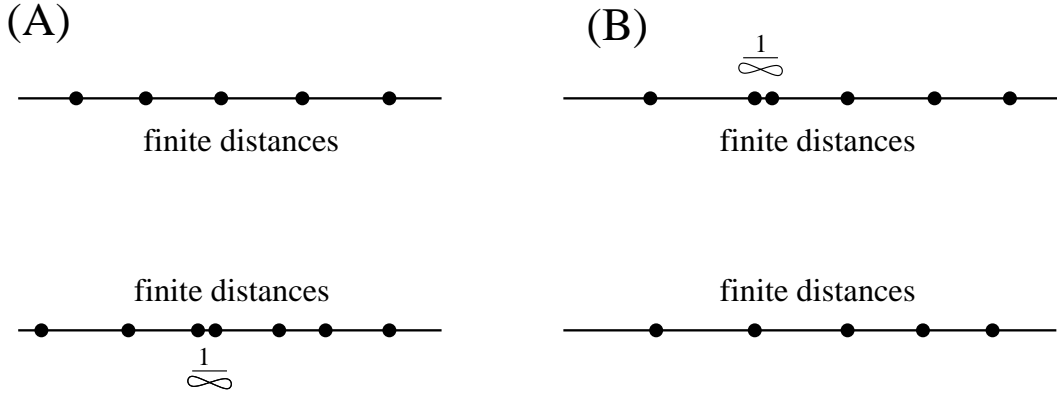


Figure 9: Non-boundary terms in the Gerstenhaber-Schack differential

The non-boundary terms (the second lines in the right hand-sides of the formulas for the Gerstenhaber-Schack differential) are corresponded to the strata drawn in Figure 9(A) for $(d_{GS})_1$ and in Figure 9(B) for $(d_{GS})_2$. (See Figure 2 above for the picture of a

general stratum of codimension 1). In the Figure 9(A) the stratum is $K(m, n) \times K(2, 1)$. The space $K(2, 1)$ is a point, and we take the function (zero-form) 1 as the corresponding element in $B^{\bullet\bullet}(2, 1)$. Analogously for Figure 9(B).

The boundary terms (the first and the third lines in the r.h.s. of the formulas) of the Gerstenhaber-Schack differential are drawn in Figure 10 (for $(d_{GS})_1$) and Figure 11 (for $(d_{GS})_2$) below. In the Figure 10 (A),(B) it is shown the two boundary terms (the

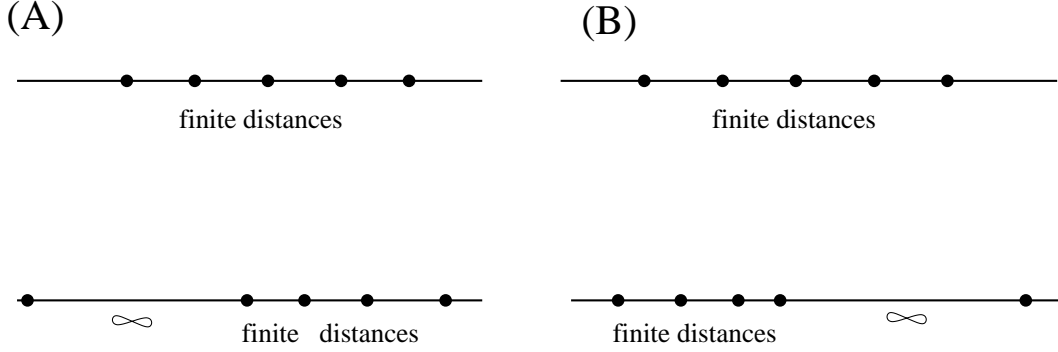


Figure 10: The two boundary terms in $(d_{GS})_1$

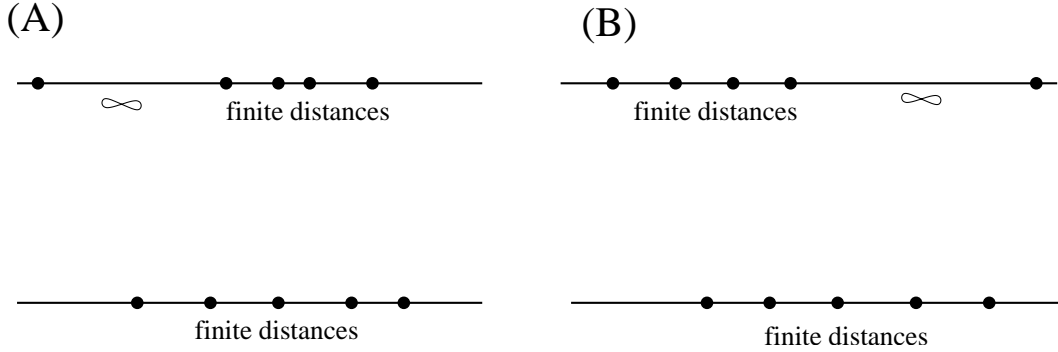


Figure 11: The two boundary terms in $(d_{GS})_2$

first and the third lines, correspondingly) of the formula (52). The boundary stratum for Figure 10(A) is $K_{1,m}^n \times K_2^{1,1,\dots,1}$. The second factor is 0-dimensional. We need to construct a form on $K_{1,m}^n$ to define the composition with values in $B^{\bullet\bullet}$. The canonical map $K_{1,m}^n \rightarrow K(m, n)$ is an isomorphism, and we just take the pull-back of the form ω on $K(m, n)$ where $\tilde{\Psi} = \Psi \otimes \omega$ and we compute $(d_{GS})_1(\tilde{\Psi})$. Analogously for the Figure 10(B), and for the Figure 11(A),(B).

We see that we can rewrite some of the summands in the first line of (50) to distinguish

the Gerstenhaber-Schack differential among these terms. Roughly speaking, these are exactly the strata of codimension 1 such that they are the products of two factors one of which is a point.

Now we are passing to the proof of the Formality Theorem.

3.4.2 We begin to prove the Formality Theorem

It follows from formulas (48) and (49)

$$\sum_{\Gamma \in \Gamma_{m,n;s}^{m+n+3s-4}} \Phi_{\Gamma}(\gamma_1, \dots, \gamma_s) \cdot \left(\int_{\partial K(m,n)} \bar{p}_* \Omega_{\Gamma} + \int_{K(m,n)} \bar{p}|_{\partial F_*} \Omega_{\Gamma} \right) = 0 \quad (54)$$

for fixed $m, n, s, \gamma_1, \dots, \gamma_s$.

We claim that the first summand of (54) is equal to the first summand of (50), and the second summand of (54) is equal to the second summand of (50). At first, by the dimensional reasons only the strata of codimension 1 do contribute to (54). First concern on the first summand.

Consider a boundary stratum σ in $\overline{K(m,n;s)}$ of the type S2 (see Figure 7). There are γ_i 's in the inner points. We should prove that the corresponding graphs are of a special form which give exactly

$$\int_{\sigma} \frac{1}{s!} \sum_{\{i_k\} \in \Sigma_s} \sum_{\{w_j\}} \pm \left(\frac{\mathcal{U}_{w_1}(\gamma_{i_1}, \dots, \gamma_{i_{w_1}}) \dots \mathcal{U}_{w_{\ell}-w_{\ell-1}}(\gamma_{i_{w_{\ell-1}+1}}, \dots, \gamma_{i_{w_{\ell}}})}{\mathcal{U}_{w_{\ell+1}-w_{\ell}}(\gamma_{i_{w_{\ell}+1}}, \dots, \gamma_{i_{w_{\ell+1}}}) \dots \mathcal{U}_{s-w_{\ell}+\ell'-1}(\gamma_{i_{w_{\ell}+\ell'-1}+1}, \dots, \gamma_{i_s})} \right)_{\sigma} \quad (55)$$

We mean that the differential forms corresponded to other graphs give 0. Let us first draw the graphs which give (55).

Consider for simplicity the case of $\overline{K(2,2;s)}$. Schematically, the picture is shown in Figure 12 below. Here in the picture the distances between the points on the boundary lines are (a bit informally speaking) infinite, and the inner points are close to the 4 external points. So, the inner points form the 4 groups (relatively to which point on the boundary it is closed). Between points of each group there is a "inner life", it means that there are some edges between the points inside each group. We show by the thin lines the only edges (oriented) between the points in different groups. They are all oriented from the top to the bottom. This picture is very good to show what kind of edges we have, but it is little bit informal. When we are interesting which distances are ∞ and which are finite, Figure 13 below is better. Here the two points **A** and **B** are in a finite distance from each other, and the distance between the two upper boundary points is ∞ . The inner points at a finite domain are divided into 2 groups (the triangles **T₁** and **T₂** in Figure 13). **There are no edges from a point of a one triangle to a point of another.** There are also inner points infinitely close to the boundary points **A** and **B** (the distances between them are of order $\frac{1}{\infty}$). There are some edges between these points. When we apply the infinite transform $(0, 0, \infty) \in G^{(3)}$ to Figure 13, the

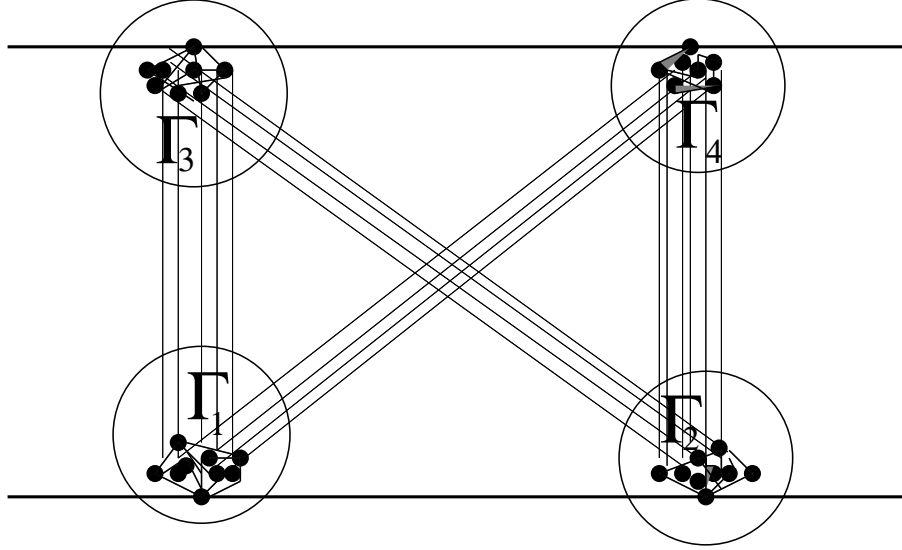


Figure 12: A typical graph which contribute to the boundary strata of the second type

configuration will be "symmetric": the upper line will look as the lower line before the transform, and vice versa. The picture in Figure 12 shows better what kind of edges we have.

Compute the weight $\int_{\sigma} \bar{p}_* \Omega_{\Gamma}$ for σ and Γ shown in Figures 12,13. *It is clear that the integral factorizes into the product of 4 integrals.* The graph Γ has $m + n + 3s - 4$ edges (here in the example $m = n = 2$), and each factor in the product is the weight of a graph Γ_i with $m_i + n_i + 3s_i - 3$ edges ($i = 1 \dots 4$). These graphs Γ_i are admissible graphs, which contribute to \mathcal{U}_i in formula (55). Now we should prove that the corresponding operator Φ_{Γ} is equal to $\frac{\Psi_1 \Psi_2}{\Theta_1 \Theta_2}$ where Ψ_1 is Φ_{Γ_1} where Γ_1 is the graph inside the triangle \mathbf{T}_1 in Figure 13, $\Psi_2 = \Phi_{\Gamma_2}$ where Γ_2 is the graph inside \mathbf{T}_2 , and Θ_1 and Θ_2 are Φ_{Γ_3} and Φ_{Γ_4} where to see Γ_3 and Γ_4 we should first apply to Figure 13 the infinite transform $(0, 0, \infty) \in G^{(3)}$. It follows from the definition of Φ_{Γ} .

Now we have the following lemma:

Lemma. *Each graph Γ which contributes to the first summand of (54) factorizes as is shown in Figures 12,13 into "disjoint union" of 4 graphs (that is, there are no "bad edges" shown in Figure 13).*

Proof. Consider an edge connecting a point from the triangle \mathbf{T}_1 with a point from the triangle \mathbf{T}_2 (see the "bad edge" on Figure 13). Consider the corresponding point of the 3-dimensional Eye (see Figure 5) corresponding to this oriented pair of points. It is clear that this point belongs to the boundary of the tetrahedron, but not to the closed interval ($\lambda = 0$). Then the value of the Propagator 2-form on this edge is 0

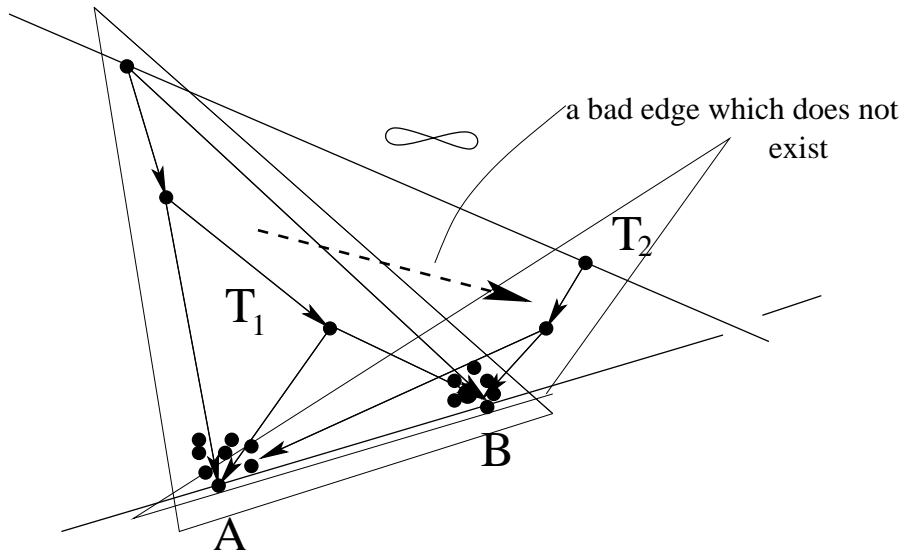


Figure 13: A better picture for a stratum of the type S2

by the construction of the Propagator in Section 2.2.2. All other graphs belong to the "factorizable" pictures. \square

We considered here the case when $m = n = 2$. The general case is absolutely analogous.

It remains to consider the second summand of (54). In the second summand of (54) we have the boundary strata of $\overline{K}(mn; s)$ of the first type (see Section 2.3). It is clear that the strata S1.2 do not contribute to the integral: as in the lemma above, the corresponding "limit" edges belong to the boundary of the Tetrahedron but not to the interval ($\lambda = 0$), and the Propagator vanishes on them. If there is no edge connecting the limit point to the finite configuration, the integral vanishes by the dimensional reasons.

There remain the strata S1.1 when several inner points move close to each other. We claim that only the case when two inner points move close to each other and are connected by a single edge does contribute. It follows from the "Kontsevich lemma in dimensions ≥ 3 which we need in dimension 3. It is true also in dimension 2 but the proof (given in [K1]) is more complicated. In dimensions ≥ 3 this lemma was found by Kontsevich in his study of the Chern-Simons theory (see, e.g. [K4]) but we do not know any place where the proof is written. Below we reproduce the original Kontsevich's proof.

3.4.3 Kontsevich Lemma in dimension ≥ 3

Consider the configuration space $\text{Conf}_n(\mathbb{R})$ defined as follows:

$$\text{Conf}_n(\mathbb{R}^d) = \{(p_1, \dots, p_n \in \mathbb{R}^d), p_i \neq p_j \text{ for } i \neq j\} \quad (56)$$

There is a $(d+1)$ -dimensional group $G(d)$ acting on the space $\text{Conf}_n(\mathbb{R}^d)$. This is the group of all d linearly independent shifts and 1-parametric family of dilatations. Consider the quotient

$$C_n(\mathbb{R}^d) = \text{Conf}_n(\mathbb{R}^d)/G(d) \quad (57)$$

If $n \geq 2$ it is a smooth manifold of dimension $nd - (d+1)$. For any two points $\{p_i, p_j\}$ there is the restriction map $t_{ij}: C_n(\mathbb{R}^d) \rightarrow C_2(\mathbb{R}^d) \simeq S^{d-1}$. Consider the volume form Ω on the sphere S^{d-1} . Denote the pull-back $t_{ij}^* \Omega$ by Ω_{ij} .

Lemma. *Let $d \geq 3$. Consider an (oriented) graph Γ with n vertices and e edges such that $e(d-1) = nd - d - 1$. Associate with Γ a top degree form Ω_Γ on $C_n(\mathbb{R}^d)$:*

$$\Omega_\Gamma = \bigwedge_{(ij) \in E_\Gamma} \Omega_{ij} \quad (58)$$

(the order in the wedge product is irrelevant) where E_Γ is the set of the edges of Γ . Then the integral $\int_{C_n(\mathbb{R}^d)} \Omega_\Gamma$ always converges and is nonzero only in the case $n = 2, e = 1$.

Proof. We prove that when $d \geq 3$ there exists at least 1 vertex of Γ of the valence ≤ 2 (the valence is defined for Γ as for an non-oriented graph). Indeed, suppose that the valences of the all vertexes are ≥ 3 . Then the number of edges $\#E_\Gamma \geq \frac{3n}{2}$. By our assumption, $e(d-1) = nd - d - 1$, therefore, $nd - d - 1 \geq (\frac{3n}{2})(d-1)$. The last inequality has no solutions for $d \geq 3$.

Denote by v a vertex of valence ≤ 2 . Consider independently the 3 cases when it is 0, 1, 2 and prove in all cases that $\int_{C_n(\mathbb{R}^d)} \Omega_\Gamma = 0$.

If the valence of v is 0 then the integral is 0 by dimensional reasons because it is an integral of a form of degree $nd - d - 1$ over a space of dimension $nd - 2d - 1$. Analogously, when the valence of v is equal to 1, put another endpoint of this edge to a fixed point (we can do it using the group $G(d)$). Then the integral by v is an integral of $(d-1)$ -form by \mathbb{R}^d which gives 0.

Consider the case when the valence of v is 2. We want to prove that $\int_{C_n(\mathbb{R}^d)} \Omega_\Gamma = 0$. Consider the symmetric integral over v' (see Figure 14). The sign of the transform $\vartheta: v \mapsto v'$ is $(-1)^d$. Suppose the points **1** and **2** are fixed and we integrate only over v . We have:

$$\int_{v \in \mathbb{R}^d} \Omega_{1v} \wedge \Omega_{2v} = (-1)^d \int_{v' \in \mathbb{R}^d} \Omega_{1v} \wedge \Omega_{2v} \quad (59)$$

We have: $\Omega_{1v'} = (-1)^d \Omega_{2v}$ and $\Omega_{2v'} = (-1)^d \Omega_{1v}$. Then we have:

$$\int_{v \in \mathbb{R}^d} \Omega_{1v} \wedge \Omega_{2v} = (-1)^d \int_{v' \in \mathbb{R}^d} \Omega_{2v'} \wedge \Omega_{1v'} \quad (60)$$

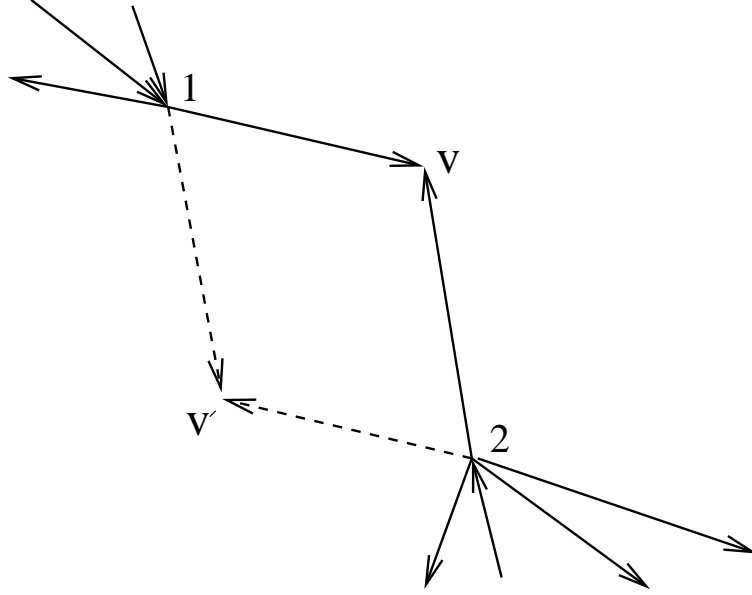


Figure 14: The proof of the Kontsevich lemma

But $\Omega_{2v'} \wedge \Omega_{1v'} = (-1)^{(d-1)^2} \Omega_{1v'} \wedge \Omega_{2v'}$, and the integral equal to itself multiplied on $(-1)^{d+(d-1)^2}$ which is equal to -1 for all d . Therefore, the integral is 0. \square

We finish to prove Theorem 3.4. In the second summand of (54) only the boundary strata of type S1.1 contribute. The integral factorizes to the product of two integrals one of them is an integral of the type of Kontsevich lemma (in dimension 3). This lemma claims that the only case which contributes to the integral is the case when two inner points connected by a 1 edge move close to each other. It gives exactly the second summands in (50).

Theorem 3.4 is proven. \square

3.4.4 The "Formality Conjecture"

Consider the bicomplex $B^{\bullet\bullet}(m, n)$. As it was noticed before, it has the only cohomology in degree 0 which is 1-dimensional. There is a very natural map

$$\int : B^{\bullet\bullet}(m, n) \rightarrow \mathbb{C}[0] \quad (61)$$

which is a quasi-isomorphism. Namely, let $\Omega = \omega_{m+n-3} + \omega_{m+n-4} + \cdots + \omega_1 + \omega_0$ be a general element of $B^{\bullet\bullet}(m, n)[0]$ where ω_i is a linear combination of *top degree* forms on

strata of codimension i . Define now the map \int as

$$\int(\Omega) = \int_{K(m,n)} \omega_{m+n-3} + \int_{\text{codim}\sigma=1} \omega_{m+n-4} + \dots + \int_{\text{codim}\sigma=m+n-4} \omega_1 + \int_{\text{codim}\sigma=m+n-3} \omega_0 \quad (62)$$

and define the map \int as 0 on $B^{\bullet\bullet}(m,n)[k]$ for $k \neq 0$. It follows immediately from the Stokes formula that the map \int vanishes on the boundaries in $B^{\bullet\bullet}(m,n)$, and it defines a quasi-isomorphism.

It motivates the following "Formality Conjecture":

Conjecture 3. *There exist the components $\tilde{\mathcal{U}}_\Gamma := \text{Alt}_\gamma \Phi_\Gamma(\gamma_1, \dots, \gamma_s) \otimes (\bar{p}_* \Omega_\Gamma + D\tilde{\Omega}_\Gamma)$ where D is the differential in $B^{\bullet\bullet}$ (that is, $\tilde{\mathcal{U}}_\Gamma$ differs from \mathcal{U}_Γ on a boundary in $B^{\bullet\bullet}$) and the corresponding $\tilde{\mathcal{U}}_s = \sum_{m,n} \sum_{\Gamma \in \Gamma_{m,n;s}} \tilde{\mathcal{U}}_\Gamma$ such that the following "formality on the level of complexes" holds (here we do not fix m, n , only fix s):*

$$\begin{aligned} & \sum_{\text{codim}\sigma=1} N_{\{w_i\}} \sum_{\{i_k\} \in \Sigma_s} \sum_{\{w_j\}} \pm \left(\frac{\tilde{\mathcal{U}}_{w_1}(\gamma_{i_1}, \dots, \gamma_{i_{w_1}}) \dots \tilde{\mathcal{U}}_{w_\ell - w_{\ell-1}}(\gamma_{i_{w_{\ell-1}+1}}, \dots, \gamma_{i_{w_\ell}})}{\tilde{\mathcal{U}}_{w_{\ell+1}-w_\ell}(\gamma_{i_{w_\ell+1}}, \dots, \gamma_{i_{w_{\ell+1}}}) \dots \tilde{\mathcal{U}}_{s-w_{\ell'+1}-1}(\gamma_{i_{w_{\ell'+1}+1}}, \dots, \gamma_{i_s})} \right)_\sigma \\ & + \sum_{1 \leq i < j \leq s} \pm \tilde{\mathcal{U}}_{s-1}(\{\gamma_i, \gamma_j\}, \gamma_1, \dots, \hat{\gamma}_i, \dots, \hat{\gamma}_j, \dots, \gamma_s) = 0 \end{aligned} \quad (63)$$

The meaning of this Conjecture is that if we would "add" some other terms to our structure depending on the strata of codimension 1 to \tilde{K}_{GS} to have a "pure" L_∞ structure, the Conjecture above would express that this structure is formal, that is, is equivalent to its cohomology $\mathfrak{g}_1^\bullet = \oplus_{m,n \geq 1} \text{Hom}(\wedge^m(V), \wedge^n(V))[-m-n+2]$. In the Subsection below we show how the Lie bracket on \mathfrak{g}_1^\bullet (which is a Poisson Lie bracket) is hidden in the operations corresponding to the strata of codimension 1 in $\tilde{K}(m,n)$.

3.4.5 The strata of codimension 1 and the Lie bracket on \mathfrak{g}_1^\bullet

First of all, formulate the following result:

Lemma. (i) *For any (co)associative bialgebra, the Gerstenhaber-Schack differential is indeed a differential, that is, $((d_{GS})_1 + (d_{GS})_2)^2 = 0$,*

(ii) *in the case of the free commutative cocommutative (co)associative bialgebra $A = S(V^*)$ where V is a finite-dimensional vector space, the Gerstenhaber-Schack cohomology of $\mathfrak{g}_2^\bullet = K_{GS}^\bullet$ is isomorphic to $\mathfrak{g}_1^\bullet = \oplus_{m,n \geq 1} \text{Hom}(\wedge^m(V), \wedge^n(V))[-m-n+2]$,*

(iii) *the following analog of the Hochschild-Kostant-Rosenberg map $\varphi: \mathfrak{g}_1^\bullet \rightarrow \mathfrak{g}_2^\bullet$ (in the case $A = S(V^*)$) defines an isomorphism on cohomology where the map φ is the operation Φ_Γ corresponding to the following graph Γ :*

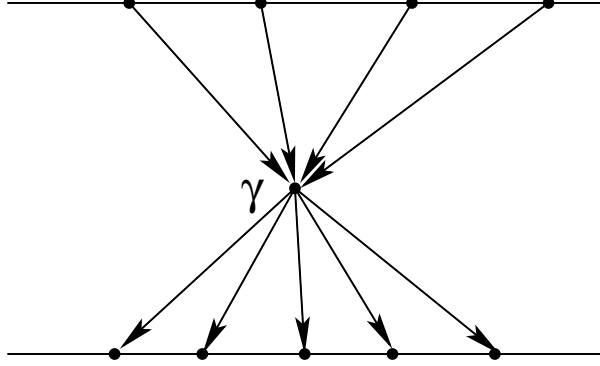


Figure 15: The analog of the Hochschild-Kostant-Rosenberg map

Proof. We just sketch the proof here. We need to compute the Gerstenhaber-Schack cohomology in the case of bialgebra $A = S(V^*)$. The two components of the Gerstenhaber-Schack differential form a bicomplex. We can use any of the two canonical spectral sequences to compute the cohomology. Compute first the cohomology of the differential $(d_{GS})_1$. We have the Hochschild cohomological complexes here $\text{Hoch}^\bullet(A, M_k)$ where $M_k = A^{\otimes k}$, and the bimodule structure is given by the left and the right multiplications on $\Delta^{k-1}(a)$. First we need to compute this Hochschild cohomology (for all k). For this we need a more direct description of the bimodule M_k . For any Lie algebra \mathfrak{g} consider the left action of \mathfrak{g} on the universal enveloping algebra $U(\mathfrak{g})$. Consider the tensor product of k of such \mathfrak{g} -modules. Then we obtain the corresponding $U(\mathfrak{g})$ -module structure on $U(\mathfrak{g})^{\otimes k}$. In the case of Abelian Lie algebra $\mathfrak{g} = V^*$ this structure on $S(V^*)^{\otimes k}$ coincides with the left action on our bimodule. The analogous construction works as well for the right action. Therefore, we need to decompose the \mathfrak{g} -(bi)module $U(\mathfrak{g})^{\otimes k}$, and then we get automatically a decomposition of the corresponding $U(\mathfrak{g})$ -(bi)module. It is easy to do in the case of an Abelian Lie algebra \mathfrak{g} (for $\mathfrak{g} = V^*$). Namely, $S(V^*)^{\otimes k}$ is the algebra of polynomials $\{f(v_1, \dots, v_k)\}$ where $v_i \in V$. The action of V^* is $f \mapsto (\xi(v_1) + \dots + \xi(v_k)) \cdot f$ for $\xi \in V^*$. Now polynomials $f(v_1 + \dots + v_k)$ form a submodule (with respect to the both left and right actions) which is isomorphic to the bimodule $A = S(V^*)$. It is clear then that as an A -bimodule, $A^{\otimes k} = A \otimes A^{\otimes k-1}$ where in the bimodule structure A acts (tautologically) only on the first factor (that is, it is a direct sum of infinitely many copies of A). We can identify $A^{\otimes k-1}$ with $\{f(v_1 - v_2, v_1 - v_3, \dots, v_1 - v_k)\}$.

Therefore, by the Hochschild-Kostant-Rosenberg theorem, the cohomology of the differential $(d_{GS})_1$ in the k -th row is $T_{poly}^\bullet(V) \otimes A^{\otimes k-1}$. Now we can compute the second differential on the image of this cohomology in K_{GS}^\bullet with respect to the Hochschild-Kostant-Rosenberg map, and then we get that the second term E_2 of the spectral sequence is isomorphic to \mathfrak{g}_1^\bullet . We need to prove independently that under the map of the item (iii) of this Lemma \mathfrak{g}_1^\bullet is *imbedded* to the cohomology. \square

Now the following question arises: which operations on the (homotopical) Gerstenhaber-Schack complex \tilde{K}_{GS}^\bullet among the operations corresponded to the strata of codimension 1 in $\overline{K(m, n)}$ define on the cohomology the Poisson Lie bracket on \mathfrak{g}_1^\bullet ? Here we answer this question.

Let $\Psi \in \text{Hom}(A^{\otimes m_1+1}, A^{\otimes n_0})$ and $\Theta \in \text{Hom}(A^{\otimes m_0}, A^{\otimes n_1+1})$. Define $\tilde{\Psi} = \Psi \otimes \omega_1$ and $\tilde{\Theta} = \Theta \otimes \omega_2$ where the form ω_1 is a top degree form on the top dimensional open stratum in $\overline{K(m_1+1, n_0)}$, and the form ω_2 is a top degree form on the top dimensional open stratum in $\overline{K(m_0, n_1+1)}$. Define all other data in the operation corresponded to the stratum of codimension 1 in $\overline{K(m, n)}$ drawn in Figure 2 as $*^{m_1} \otimes 1$ and $\Delta^{n_1} \otimes 1$ when the 1's above are considered as zero degree differential forms on $\overline{K(m_1+1, 1)}$ and on $\overline{K(1, n_1+1)}$ correspondingly. Then the composition

$$\frac{\tilde{\Psi}_1 \tilde{\Psi}_2 \dots \tilde{\Psi}_{n_1+1}}{\tilde{\Theta}_1 \tilde{\Theta}_2 \dots \tilde{\Theta}_{m_1+1}} := \frac{\Psi_1 \Psi_2 \dots \Psi_{n_1+1}}{\Theta_1 \Theta_2 \dots \Theta_{m_1+1}} \otimes [\omega]_\sigma \quad (64)$$

and the form ω in the r.h.s is a *top degree form* on the stratum σ in $\overline{K(m, n)}$.

Even if we suppose that this operation when ω_1 and ω_2 are not top degree forms is zero, it is well-defined (the latter means it is compatible with the differential in $B^{\bullet\bullet}$).

We claim that on the cohomology (of both differentials in $B^{\bullet\bullet}$ and of the Gerstenhaber-Schack differential) this operation defines exactly the Poisson Lie bracket in \mathfrak{g}_1^\bullet (when we take the sum over all σ with fixed m_0, m_1, n_0, n_1). Moreover, our Hochschild-Kostant-Rosenberg map is a "Lie algebra map" up to a boundary. This is an initial point for the construction of the "formality" morphism.

3.5 Deformation quantization of Lie bialgebras

Let V be a Lie bialgebra. Recall that it means that we have $\alpha \in \text{Hom}(\wedge^2 V, V)$ and $\beta \in \text{Hom}(V, \wedge^2 V)$ such that $\{\alpha, \alpha\} = 0$, $\{\beta, \beta\} = 0$, and $\{\alpha, \beta\} = 0$ (where $\{\cdot, \cdot\}$ denotes the Poisson Lie bracket in \mathfrak{g}_1^\bullet).

Define a product and a coproduct on $S(V^*)$ as follows:

$$f * g = \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1!} \frac{1}{\ell_2!} \sum_{\Gamma \in \Gamma_{2,1;\ell_1+\ell_2}^{3(\ell_1+\ell_2)}} \hbar_1^{\ell_1} \hbar_2^{\ell_2} \mathcal{U}_\Gamma(\alpha, \dots, \alpha, \beta, \dots, \beta)(f \otimes g) \quad (65)$$

and

$$\Delta_*(f) = \sum_{\ell_1, \ell_2 \geq 0} \frac{1}{\ell_1!} \frac{1}{\ell_2!} \sum_{\Gamma \in \Gamma_{1,2;\ell_1+\ell_2}^{3(\ell_1+\ell_2)}} \hbar_1^{\ell_1} \hbar_2^{\ell_2} \mathcal{U}_\Gamma(\alpha, \dots, \alpha, \beta, \dots, \beta)(f) \quad (66)$$

Here in the right-hand sides of the formulas we have ℓ_1 of α 's and ℓ_2 of β 's. The values of \mathcal{U}_Γ are top degree forms on $\overline{K(2, 1)}$ and $\overline{K(1, 2)}$ which are just points, and we identify the function on them with the numbers.

Theorem. *The product $f * g$ and the coproduct $\Delta_*(f)$ defined above satisfy the axioms (i)-(iii) of (co)associative bialgebras (see the first page of the Introduction).*

Proof. To prove the associativity, apply the formality theorem to the space $\overline{K(3, 1; \ell_1 + \ell_2)}$ and to $\gamma_1, \dots, \gamma_{\ell_1} = \alpha, \gamma_{\ell_1+1}, \dots, \gamma_{\ell_1+\ell_2} = \beta$.

To prove the coassociativity, apply the formality theorem to the space $\overline{K(1, 3; \ell_1 + \ell_2)}$ and to the γ_i 's as above.

To prove the compatibility, apply the formality theorem to the space $\overline{K(2, 2; \ell_1 + \ell_2)}$ and to the γ_i 's as above. \square

Notice that we obtain a 2-parametric deformation quantization of Lie bialgebras.

3.6 An informal discussion about why we have not an L_∞ structure here

Probably the most strangeness among the strange things in this paper is that the right-hand sides of our "formality" do not obey the L_∞ Jacobi identity. Here we want to explain a geometrical cause for that.

Suppose that we have a collection of open manifolds M_k and of their compactifications \overline{M}_k which all are manifold with corners such that any face of codimension ℓ is a product of $\ell + 1$ different M_i 's (maybe these are not the most right conditions for the description in general). Then when we compute the boundary operator ∂M_k we get a quadratic expression in M_i 's which is the sum over the all strata of codimension 1. Then suppose we have a stratum of codimension 1 of M_k which is a product $M_i \times M_j$. How to compute the boundary $\partial(M_i \times M_j)$? The most natural is to suppose that the Leibniz rule for ∂ holds. (Of course, we have in mind that the strata are labeled by some combinatorial objects like trees, and the Leibniz rule should be understood in this sense). For example, in the case of Stasheff polyhedra the Leibniz rule holds.

Suppose we have something like an "algebraic representation" of the family $\{M_i\}$. It means that we associate to each space M_k a graded vector space V_k and to each stratum of codimension 1 an operation on $\oplus_k V_k$. Suppose that these operations are compatible with the boundary operations. The latter means that if a stratum σ_1 can be obtained by a "degeneration of codimension 1" from a stratum σ , and a stratum σ_2 can be obtained by a degeneration of codimension 1 from a stratum σ_1 , then it implies that the corresponding compositions can be also obtained functorially one from another. *Then the Jacobi identity is an immediate consequence of the Leibniz rule for the boundary operator ∂ .* This is more or less in the spirit of the Markl's paper [M1]. It follows from the identity $\partial^2 = 0$ for the boundary operator.

But we can imagine the situation when the Leibniz rule is not satisfied. For instance, it could be an operator of the second order. It means that to compute the boundary of any stratum we need to know the boundaries of strata of codimension 0 and of strata of codimension 1. Then the Jacobi identity will be replaced by a more complicated

structure. Probably, it will be an odd vector field Q on $L[1]$ such that $Q^3 = 0$ (not $Q^2 = 0$ as in the L_∞ -case).

In our situation the problem is that the Leibniz rule is not satisfied. Let us explain how it works:

Consider the space $\overline{K(m, n)}$ constructed in the paper. We canonically identify a boundary stratum σ of codimension 1 with a product of two spaces of different type, $K_{m_1+1}^{1, \dots, 1, n_0, 1, \dots, 1} \times K_{1, \dots, 1, m_0, 1, \dots, 1}^{n_1+1}$. Then we compute the boundary of σ of codimension 1. Probably, for the spaces $K_{m_1+1}^{1, \dots, 1, n_0, 1, \dots, 1}$ and $K_{1, \dots, 1, m_0, 1, \dots, 1}^{n_1+1}$ the Leibniz rule is satisfied. The map $i: K_{m_1+1}^{1, \dots, 1, n_0, 1, \dots, 1} \times K_{1, \dots, 1, m_0, 1, \dots, 1}^{n_1+1} \rightarrow \overline{K(m, n)}$ is an imbedding. But the canonical extension $\overline{K_{m_1+1}^{1, \dots, 1, n_0, 1, \dots, 1} \times K_{1, \dots, 1, m_0, 1, \dots, 1}^{n_1+1}} \rightarrow \overline{K(m, n)}$ is *not* an imbedding. In particular, it maps the boundary of codimension 1 to the boundary of σ of codimension 1 surjectively, but it contracts some components to points. Therefore, the Leibniz rule is satisfied modulo these components.

It is very interesting to understand the whole structure arising from this construction.

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P.S. The author afraids that some fractions with factorials in the formulas concerning the "formality" are computed wrong :)

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